AN INVERSE SOURCE PROBLEM FOR A ONE DIMENSIONAL SPACE-TIME FRACTIONAL DIFFUSION EQUATION

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Abstract. Fractional(nonlocal) diffusion equations replace the integer-order derivatives in space and time by their fractional-order analogues and they are used to model anomalous diffusion, especially in physics. This paper deals with a nonlocal inverse source problem for a one dimensional space-time fractional diffusion equation $\partial_t^\beta u = -r^\beta(-\Delta)^{\alpha/2}u(t,x) + f(x)h(t,x)$ where $(t,x) \in \Omega_T := (0,T) \times \Omega$ and $\Omega = (-1,1)$. At first we define and analyze the direct problem for the space-time fractional diffusion equation. Later we define the inverse source problem. Furthermore, we set up an operator equation $A_r f(x) + \Theta(x) = f(x)$ and derive the relation between the solutions of the operator equation and the inverse source problem. We also prove some important properties of the operator A_r. By using these properties and analytic Fredholm theorem, we prove that the inverse source problem is well-posed, i.e, $f(t,x)$ can be determined uniquely and depends continuously on additional data $u(T,x), x \in \Omega$.

1. Introduction

In this paper, we consider an initial-boundary value problem for a space-time fractional equation

$$(1.1)\quad \begin{cases} \dfrac{\partial^\beta}{\partial t^\beta}u(t,x) = -r^\beta(-\Delta)^{\alpha/2}u(t,x) + f(x)h(t,x), (t,x) \in \Omega_T, \\ u(t,-1) = u(t,1) = 0,\ 0 < t < T, \\ u(0,x) = 0, x \in \Omega, \end{cases}$$

where $\Omega_T := (0,T) \times \Omega$, $\Omega = (-1,1)$, $r > 0$ is a parameter, $f(x) \in L_2(\Omega)$, $h(t,x) \in C^1([0,T]; L^\infty(\Omega))$ are given functions, $\beta \in (0,1)$, $\alpha \in (1,2)$ are fractional order of the time and the space derivatives respectively and $T > 0$ is a final time.

Key words and phrases. Fractional derivative; Fractional Laplacian; Inverse Source problem; Mittag-Leffler function.

The fractional-time derivative considered here is the Caputo fractional derivative of order $0 < \beta < 1$ and is defined by

$$\frac{\partial^\beta f(t)}{\partial t^\beta} := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial f(r)}{\partial r} \frac{dr}{(t-r)^\beta}, \tag{1.2}$$

where Γ is the Gamma function. This was intended to properly handle initial values [3, 4, 6], since its Laplace transform(LT) $s^\beta \tilde{f}(s) - s^{\beta-1} f(0)$ incorporates the initial value in the same way as the first derivative. Here, $\tilde{f}(s)$ is the usual Laplace transform. It is well-known that the Caputo derivative has a continuous spectrum [4], with eigenfunctions given in terms of the Mittag-Leffler function

$$E_\beta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

In fact, it is easy to see that, $f(t) = E_\beta(-\lambda t^\beta)$ solves the eigenvalue equation

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = -\lambda f(t), f(0) = 1$$

for any $\lambda > 0$. This is easily verified by differentiating term-by-term and using the fact that t^p has Caputo derivative $t^{p-\beta} \dfrac{\Gamma(p+1)}{\Gamma(p+1-\beta)}$ for $p > 0$ and $0 < \beta \leq 1$. $0 < \beta < 1$ is taken for slow diffusion, and is related to the parameter specifying the large-time behavior of the waiting-time distribution function, see [13] and some of the references cited therein.

For $0 < \alpha < 2, (-\Delta)^{\alpha/2} u$ denotes the fractional Laplacian of u. It turns out that it is easier to define it by using the spectral decomposition of the Laplace operator: We take $\{\bar{\lambda}_k, \psi_k\}$ the eigenvalues and corresponding eigenvectors of the Laplacian operator in Ω with Dirichlet boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta \psi_k = \bar{\lambda}_k \psi_k, & \text{in} \quad \Omega, \\ \psi_k = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

We then define the operator $(-\Delta)^{\alpha/2}$ by

$$(-\Delta)^{\alpha/2} u := \sum_{k=0}^{\infty} c_k \psi_k(x) \mapsto -\sum_{k=0}^{\infty} c_k \bar{\lambda}_k^{\alpha/2} \psi_k(x),$$

which maps $H_0^\alpha(\Omega)$ onto $L^2(\Omega)$.

If f is C^1-function on $[0,\infty)$ satisfying $|f'(t)| \leq Ct^{\gamma-1}$ for some $\gamma > 0$, then by (1.2), the Caputo derivative $\dfrac{\partial^\beta f(t)}{\partial t^\beta}$ of f exists for all $t > 0$ and the derivative is continuous in $t > 0$. Kilbas et al [9] and Podlubny [13] can be referred for further properties of the Caputo derivative.

In (1.1) the term $f(x)h(t,x)$ models a source term, and it is important to determine $f(x)$ for realizing observation data. That is to say our main goal in this paper is: Let $r > 0$ be fixed. Determine $u(t,x) = u(r,f)(t,x)$ and $f(x)$ for $t \in (0,T)$ and $x \in \Omega$ satisfying (1.1) and

$$u(T,x) = \varphi(x), \quad x \in \bar{\Omega}. \tag{1.3}$$

This inverse problem is by final overdetermining data. We prove that the inverse problem is well-posed in the sense of Hadamard except for a finite set of $r > 0$. Our main tools are the representation formula of the solution to (1.1), the theory of analytic perturbation of linear operators(see e.g. [8]) and the uniqueness in the inverse problem (1.1-1.3). For similar studies one can refer to [15], [16], [17] and some of the references cited therein. The main difference from these studies is that the current study includes both space and time fractional derivatives.

This paper is organized as follows: In the next section formulation of the direct and inverse problems along with their analyses are presented. Section 3 includes some properties of the operator A_r and the final section of the paper includes both the statement and the proof of the main result of the paper.

2. Formulations and analyses of the direct and inverse problems

In this section, we define the direct and the inverse source problems. For given inputs β, α, $f(x)$ and $h(t,x)$, the problem (1.1) is called the direct (forward) problem. Throughout this paper, we assume the following condition on $h(t,x)$

$$\big|h(T,x)\big| \geq \delta > 0, x \in \overline{\Omega}. \tag{2.1}$$

Like most direct problems of the mathematical physics, the problem (1.1) is well-posed, see the very recent interesting paper [10] for details.

The formal solution to the direct problem (1.1) is given in the form (see [12] for details)

(2.2)
$$u(t,x)=\sum_{n=1}^{\infty}\left(\int_0^t \tau^{\beta-1}E_{\beta,\beta}\big(-\lambda_n r^{\beta}\tau^{\beta}\big)\Big\langle f(x)h(t-\tau,x),\psi_n(x)\Big\rangle d\tau\right)\psi_n(x),$$

where $\lambda_n=\big(\bar{\lambda}_n\big)^{\alpha/2}$, $\bar{\lambda}_n$ and $\{\psi_n\}_{n\geq 1}$ are eigenvalues and eigenvectors of the classical Laplace operator $-\Delta$ respectively, i.e, $-\Delta\psi_n=\bar{\lambda}_n\psi_n$. A simple calculation yields $\bar{\lambda}_n=\dfrac{n^2\pi^2}{4}$ hence $\lambda_n=(\dfrac{n\pi}{2})^{\alpha}$ with $\psi_n(x)=\sin\left(\dfrac{n\pi x}{2}\right)$ when n is even and $\psi_n(x)=\cos\left(\dfrac{n\pi x}{2}\right)$ when n is odd. $\big\langle\cdot\,,\cdot\big\rangle$ denotes the standard inner product on $L^2(\Omega)$ and $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function defined as follows

$$E_{\alpha,\beta}(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, z\in\mathbb{C},$$

where $\alpha>0$ and $\beta\in\mathbb{R}$ are arbitrary constants, Γ is the Gamma function. We note that $\{\bar{\lambda}_n\}_{n\geq 1}$ is a sequence of positive numbers $0<\bar{\lambda}_1\leq\bar{\lambda}_2\leq\cdots$, $\{\psi_n\}_{n\geq 1}$ is an orthonormal basis for $L_2(\Omega)$. It is proved in [12] that (2.2) is the generalized solution to the problem (1.1) that can be interpreted as the solution in the classical sense under certain additional conditions. For practical use in latter sections, if we use the substitution $\tau^{\star}=t-\tau$ (later replace $\tau^{\star}$ by τ) in (2.2), we get the following useful formula for the solution of (1.1)

$$\begin{aligned}u(t,x)=\sum_{n=1}^{\infty}\bigg(\int_0^t&(t-\tau)^{\beta-1}E_{\beta,\beta}\big(-\lambda_n r^{\beta}(t-\tau)^{\beta}\big)\\&\times\Big\langle f(x)h(\tau,x),\psi_n(x)\Big\rangle d\tau\bigg)\psi_n(x).\end{aligned}\tag{2.3}$$

The following theorems indicate some important properties of the Mittag-Leffler function, (see the formula (1.83) and Theorem 1.6 in [13] respectively) which provide technical convenience in ensuing theorems ahead.

Lemma 2.1. *If $\alpha<2$, β is arbitrary real number, μ is such that $\dfrac{\pi\alpha}{2}<\mu<\min\{\pi\alpha,\pi\}$, $\mu\leq|\arg(z)|\leq\pi$, and C_0 is a real constant, then*

$$|E_{\alpha,\beta}(z)|\leq\frac{C_0}{1+|z|}.$$

Lemma 2.2. *The following equality holds for $\lambda > 0$, $\alpha > 0$ and $m \in \mathbb{N}$*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \; t > 0.$$

Now we show that the series on the right-hand side of (2.3) is convergent uniformly in $x \in \Omega$ and $t \in (0, T]$. By using Theorem 2.1, Cauchy-Schwarz inequality, the analytical formula of $\psi_n(x)$, the assumption $f \in L_2$, boundedness of the function $h(t, x)$, and the substitution $\tau = t(1-s)$ in the integral appearing at the third step we get

$$\begin{aligned}
&\left| \sum_{n=1}^{\infty} \left(\int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(-\lambda_n r^\beta (t-\tau)^\beta) \langle f(x)h(\tau,x), \psi_n(x) \rangle d\tau \right) \psi_n(x) \right| \\
&\le C_0 \sum_{n=1}^{\infty} \left| \int_0^t \frac{(t-\tau)^{\beta-1}}{1+\lambda_n r^\beta (t-\tau)^\beta} \|fh\| d\tau \right| \\
&\qquad \le C_1 \sum_{n=1}^{\infty} \left| \int_0^t \frac{(t-\tau)^{\beta-1}}{1+\lambda_n r^\beta (t-\tau)^\beta} d\tau \right| \\
&\qquad\qquad = C_1 \sum_{n=1}^{\infty} \left| \int_0^1 \frac{s^{\beta-1} t^\beta}{1+\lambda_n r^\beta s^\beta t^\beta} ds \right|.
\end{aligned} \tag{2.4}$$

Next we apply Young's inequality $ab \le \dfrac{a^p}{p} + \dfrac{b^q}{q}$ to the numerator $1 + \lambda_n r^\beta s^\beta t^\beta$ with $a = (1-\mu)^{-(1-\mu)}$, $b = \left(\lambda_n r^\beta s^\beta t^\beta\right)^\mu \mu^{-\mu}$, $p = \dfrac{1}{1-\mu}$, $q = \dfrac{1}{\mu}$ and $\mu \in (0,1)$ to arrive

$$C(\mu) \left(\lambda_n r^\beta s^\beta t^\beta\right)^\mu \le 1 + \lambda_n r^\beta s^\beta, \tag{2.5}$$

where $C(\mu) = (1-\mu)^{-(1-\mu)} \, \mu^{-\mu}$. Therefore we have

$$\sum_{n=1}^{\infty} \left| \int_0^1 \frac{s^{\beta-1} t^\beta}{1+\lambda_n r^\beta s^\beta t^\beta} ds \right| \le \sum_{n=1}^{\infty} \left| \frac{t^\beta}{C(\mu)\left(\lambda_n r^\beta t^\beta\right)^\mu} \int_0^1 \frac{s^{\beta-1}}{s^{\beta\mu}} ds \right|. \tag{2.6}$$

Since $\mu \in (0,1)$ and $\beta > 0$, the integral appearing at the right hand side of (2.6) is always convergent. Thus there exist a constant $C_2 > 0$ so that the following holds

$$\sum_{n=1}^{\infty} \left| \int_0^1 \frac{s^{\beta-1} t^\beta}{1+\lambda_n r^\beta s^\beta t^\beta} ds \right| \le C_2 \frac{t^\beta}{C(\mu)(rt)^{\beta\mu}} \sum_{n=1}^{\infty} \frac{1}{(\lambda_n)^\mu}. \tag{2.7}$$

By $\lambda_n = \left(\dfrac{n\pi}{2}\right)^{\alpha}$ and $\mu \in (0,1)$ for the series in (2.7) to be convergent α must satisfy $\alpha\mu > 1$ so that we must have $\alpha > 1$ (we take $\alpha \in (1,2)$). By (2.4), we conclude that the series on the right-hand side of (2.3) is convergent uniformly in $x \in \Omega$ and $t \in (0,T]$.

In the proof of Theorem 2.4, we employ Young's inequality for convolutions. For the sake of the completeness we recall it here in the form we need.

Lemma 2.3. [2] *Suppose that f is in $L_p(0,\infty)$, g is in $L_q(0,\infty)$, $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r} + 1$ with $1 \le p$ and $q, r \le \infty$. Then*

$$\big\|f * g\big\|_r \le \|f\|_p \big\|g\big\|_q,$$

where $f*g$ is convolution of f and g, i.e, $(f*g)(t) = \displaystyle\int_0^t f(\tau)g(t-\tau)\,d\tau$.

Now we prove the following theorem which will be useful for the proof of the main result of the paper.

Theorem 2.1. *Let $f(x) \in L_2(\Omega)$, $h(t,x) \in C^1([0,T]; L^\infty(\Omega))$. Then there exists a unique weak solution of the problem (1.1) such that $u \in L_2(0,T; H_0^\alpha(\Omega))$ and $\dfrac{\partial^\beta}{\partial t^\beta} u \in L_2((0,T)\times\Omega)$. Moreover, there exists a constant C such that the following inequality holds*

$$\big\|u\big\|_{L_2(0,T;H_0^\alpha(\Omega))} + \big\|\partial_t^\beta u\big\|_{L_2((0,T)\times\Omega)} \le C\big\|f\big\|_{L^2(\Omega)}. \tag{2.8}$$

Before the proof of this theorem, we recall the fractional Sobolev space

$$H_0^\alpha(\Omega) := \left\{ u = \sum_{n=1}^{\infty} a_n\psi_n : \|u\|^2_{H_0^\alpha} = \sum_{n=1}^{\infty} a_n^2 \bar{\lambda}_n^\alpha < +\infty \right\}, \tag{2.9}$$

with the following equivalence

$$\big\|u\big\|_{H_0^\alpha(\Omega)} = \big\|(-\Delta)^{\frac{\alpha}{2}} u\big\|_{L_2(\Omega)}. \tag{2.10}$$

Proof. The existence and uniqueness of the weak solution can be established following the arguments in [4], [12] and [15]. So we only prove (2.8).

By using Theorem 2.1, Theorem 2.2 and $E_{\beta,\beta}(-\eta) \geq 0$, $0 < \beta < 1$, we have

$$\begin{aligned}
\int_0^\eta |t^{\beta-1}E_{\beta,\beta}(-\lambda_n t^\beta)|dt &= \int_0^\eta t^{\beta-1}E_{\beta,\beta}(-\lambda_n t^\beta) \\
&= -\frac{1}{\lambda_n}\int_0^\eta \frac{d}{dt}E_{\beta,1}(-\lambda_n t^\beta)dt \\
&= \frac{1}{\lambda_n}\big(1 - E_{\beta,1}(-\lambda_n \eta^\beta)\big).
\end{aligned} \tag{2.11}$$

Following [13],(pp. 140) by means of the Laplace transform we see that

$$\begin{aligned}
&\frac{\partial^\beta}{\partial t^\beta}\int_0^t \Big\langle f(x)h(\tau,x), \psi_n(x)\Big\rangle (t-\tau)^{\beta-1}E_{\beta,\beta}\big(-\lambda_n r^\beta (t-\tau)^\beta\big)d\tau \\
&= -\lambda_n \int_0^t \Big\langle f(x)h(\tau,x), \psi_n(x)\Big\rangle (t-\tau)^{\beta-1}E_{\beta,\beta}\big(-\lambda_n r^\beta (t-\tau)^\beta\big)d\tau \\
&+ \Big\langle f(x)h(t,x), \psi_n(x)\Big\rangle.
\end{aligned} \tag{2.12}$$

From Theorem 2.3, (2.11) and (2.12) we have

$$\begin{aligned}
&\left\| \frac{\partial^\beta}{\partial t^\beta}\int_0^t \Big\langle f(x)h(\tau,x), \psi_n(x)\Big\rangle (t-\tau)^{\beta-1}E_{\beta,\beta}\big(-\lambda_n r^\beta (t-\tau)^\beta\big)d\tau \right\|_{L_2(0,T)} \\
&\quad \leq C_3 \left\| \Big\langle f(x)h(t,x), \psi_n(x)\Big\rangle \right\|_{L_2(0,T)} + C_4 \left\| \Big\langle f(x)h(t,x), \psi_n(x)\Big\rangle \right\|_{L_2(0,T)} \\
&\quad \times \left\| \lambda_n t^{\beta-1}E_{\beta,\beta}\big(-\lambda_n r^\beta t^\beta\big) \right\|_{L_1(0,T)} \leq C_5 \left\| \Big\langle f(x)h(t,x), \psi_n(x)\Big\rangle \right\|_{L_2(0,T)}.
\end{aligned} \tag{2.13}$$

By using (2.13) and the assumption $h(t,x) \in C^1([0,T]; L^\infty(\Omega))$, we get

$$\begin{aligned}
&\left\| \frac{\partial^\beta}{\partial t^\beta}u \right\|^2_{L_2((0,T)\times\Omega)} = \sum_{n=1}^\infty \int_0^T \left| \frac{\partial^\beta}{\partial t^\beta}\bigg(\int_0^t \Big\langle f(x)h(\tau,x), \psi_n(x)\Big\rangle (t-\tau)^{\beta-1} \right. \\
&\left. \times E_{\beta,\beta}\big(-\lambda_n r^\beta (t-\tau)^\beta\big)d\tau\bigg)\right|^2 dt \leq C_5 \sum_{n=1}^\infty \int_0^T \left| \Big\langle f(x), \psi_n(x)\Big\rangle \right|^2 dt \\
&\leq C_6 \big\| f \big\|^2_{L_2(\Omega)}.
\end{aligned} \tag{2.14}$$

By (2.10), we see also $\|u\|_{L_2(0,T;H_0^\alpha(\Omega))} \leq C_7 \|f\|_{L_2(\Omega)}$. This completes the proof. □

Next we define the inverse problem. The inverse problem here consists of determining the pair of the unknown functions $\Big\{u(t,x), f(x)\Big\}$ in the space-time fractional diffusion problem (1.1) from the final overdetermining data (also called additional condition or measured output data) (1.3), for the given function $h(t,x)$, the orders of time and space derivatives β, α and a fixed $r > 0$, that is, the problem (1.1), (1.3) is called the inverse problem.

Now we reformulate the inverse problem. For this purpose, we define the following operator equation

$$A_r f(x) + \Theta(x) = f(x), \tag{2.15}$$

where $A_r(f) : L_2(\Omega) \longrightarrow L^2(\Omega)$, $A_r(f)$ and $\Theta(x)$ defined by

$$A_r f(x) = \frac{\dfrac{\partial^\beta}{\partial t^\beta} u(r,f)(T,x)}{h(T,x)}, \ \Theta(x) = -\frac{r^\beta(\Delta^{\alpha/2}\varphi)(x)}{h(T,x)}, \tag{2.16}$$

respectively. Here we denote $u = u(r,f)$ to emphasize the dependence of the solution $u(t,x)$ of (1.1) to both $r > 0$ and $f(x)$. The following lemma indicates the relationship between the operator equation (2.15) and the inverse problem (1.1), (1.3).

Lemma 2.4. *Let $I \subset (0,\infty)$ and $r \in I$ be fixed. Then the operator equation (2.15) has a solution (a unique solution) $f \in L_2(\Omega)$ if and only if the inverse problem (1.1), (1.3) has a solution (a unique solution) $\Big\{u(r,f), f\Big\} \in L_2(0,T; H_0^\alpha(\Omega)) \times L_2(\Omega)$.*

Proof. First, suppose that the operator equation (2.15) has a solution $f \in L_2(\Omega)$. Then if we substitute this $f(x)$ into the one dimensional space-time fractional diffusion equation, we have a unique solution $u(r,f)$ to (1.1), see [15]. Now we show that the function $u(r,f)$ satisfies the additional condition (1.3). For this purpose, let's assume that

$$u(T,x) = \varphi_1(x), \quad x \in \Omega.$$

Since $u \in L_2(0,T; H_0^\alpha(\Omega))$, we have $\varphi_1(x) \in H_0^\alpha(\Omega)$. By (2.15), we conclude that $-r^\beta(-\Delta)^{\alpha/2}(\varphi_1 - \varphi) = 0$. Since the Fractional Laplacian operator is invertible, $r \neq 0$ and $\varphi_1(x) - \varphi(x) = 0, x \in \Omega$ we get $\varphi(x) = \varphi_1(x)$, $x \in \overline{\Omega}$. The converse of the assertion is directly seen in rewriting the space-time fractional diffusion equation in (2.15). The uniqueness part of the Lemma 2.4 is a direct result of the unique solvability of the problem (1.1) and the first part of the Lemma 2.4.

□

The next section is devoted to show some important properties of the operator A_r, defined by (2.16). These properties with Lemma 2.4 will be used to prove the main result of the paper.

3. Some properties of the operator A_r

This section is devoted to state and prove some properties of the operator A_r which are useful tools for a certain answer to the question of the unique solvability of the inverse problem (1.1), (1.3).

Theorem 3.1. *For $r \in I$, the operator $A_r : L^2(\Omega) \to L^2(\Omega)$ is a compact operator.*

Proof. By (2.2) we have

(3.1)
$$\frac{\partial^\beta}{\partial t^\beta} u(t,x) = f(x)h(t,x) - \sum_{n=1}^{\infty} \Big(\int_0^t \tau^{\beta-1} E_{\beta,\beta}\big(-\lambda_n r^\beta \tau^\beta\big) \Big\langle f(x)h(t-\tau,x), \psi_n(x) \Big\rangle d\tau \Big) r^\beta \lambda_n \psi_n(x).$$

By Theorem 2.2 and integration by parts we have

(3.2)
$$\begin{aligned}
&\int_0^t \tau^{\beta-1} E_{\beta,\beta}(-\lambda_n r^\beta \tau^\beta) \Big\langle f(x)h(t-\tau,x), \psi_n(x) \Big\rangle d\tau \\
&= \int_0^t \Big\langle f(x)h(t-\tau,x), \psi_n(x) \Big\rangle \frac{d}{d\tau} \Big(-\frac{1}{\lambda_n r^\beta} E_\beta\big(-\lambda_n r^\beta \tau^\beta\big) \Big) d\tau \\
&= \frac{1}{\lambda_n r^\beta} \Big(\Big\langle f(x)h(t,x), \psi_n(x) \Big\rangle - \Big\langle f(x)h(0,x), \psi_n(x) \Big\rangle E_\beta\big(-\lambda_n r^\beta \tau^\beta\big) \Big) \\
&+ \frac{1}{\lambda_n r^\beta} \int_0^t \Big\langle f(x) \frac{\partial}{\partial \tau} h(t-\tau,x), \psi_n(x) \Big\rangle E_\beta\big(-\lambda_n r^\beta \tau^\beta\big) d\tau.
\end{aligned}$$

If we substitute (3.2) in (3.1), we get

(3.3)
$$\begin{aligned}
\frac{\partial^\beta}{\partial t^\beta} u(t,x) &= \sum_{n=1}^{\infty} \Big(\Big\langle f(x)h(0,x), \psi_n(x) \Big\rangle E_\beta\big(-\lambda_n r^\beta t^\beta\big) \Big) \psi_n(x) \\
&- \sum_{n=1}^{\infty} \Big(\int_0^t \Big\langle f(x) \frac{\partial}{\partial \tau} h(t-\tau,x), \psi_n(x) \Big\rangle E_\beta\big(-\lambda_n r^\beta \tau^\beta\big) d\tau \Big) \psi_n(x).
\end{aligned}$$

Consequently, by using (2.9), (2.1), Cauhy-Schwarz inequality and Parseval's identity we obtain

(3.4)
$$
\begin{aligned}
&\left\| \frac{\partial^\beta}{\partial t^\beta} u(T,x) \right\|^2_{H_0^\alpha(\Omega)} = \sum_{n=1}^{\infty} \bar{\lambda}_n^\alpha \Big(\Big\langle f(x)h(0,x), \psi_n(x) \Big\rangle E_\beta\big(-\lambda_n r^\beta T^\beta\big) \\
&\quad - \int_0^T \Big\langle f(x)\frac{\partial}{\partial \tau} h(T-\tau,x), \psi_n(x) \Big\rangle E_\beta\big(-\lambda_n r^\beta \tau^\beta\big) d\tau \Big)^2 \\
&\quad \le 2\sum_{n=1}^{\infty} \bar{\lambda}_n^\alpha \Big\{ \Big\langle f(x)h(0,x), \psi_n(x) \Big\rangle^2 \Big(E_\beta\big(-\lambda_n r^\beta T^\beta\big) \Big)^2 \\
&\quad + \Big(\int_0^T \Big\langle f(x)\frac{\partial}{\partial \tau} h(T-\tau,x), \psi_n(x) \Big\rangle E_\beta\big(-\lambda_n r^\beta \tau^\beta\big) d\tau \Big)^2 \Big\} \\
&\quad \le 2\sum_{n=1}^{\infty} \bar{\lambda}_n^\alpha \Big\{ \Big(\frac{C_8}{1+\lambda_n r^\beta T^\beta} \Big)^2 \Big\langle f(x)h(0,x), \psi_n(x) \Big\rangle^2 \\
&\quad + \int_0^T \Big(\frac{C_8}{1+\lambda_n r^\beta (T-\tau)^\beta} \Big)^2 d\tau \int_0^T \Big(\Big\langle f(x)\frac{\partial}{\partial \tau} h(\tau,x), \psi_n(x) \Big\rangle \Big)^2 d\tau \Big\} \\
&\quad \le 2\sum_{n=1}^{\infty} \bar{\lambda}_n^\alpha \Big\{ \Big(\frac{C_8}{\lambda_n r^\beta T^\beta} \Big)^2 \Big\langle f(x)h(0,x), \psi_n(x) \Big\rangle^2 \\
&\quad + \int_0^T \Big(\frac{C_8}{1+\lambda_n r^\beta (T-\tau)^\beta} \Big)^2 d\tau \int_0^T \Big(\Big\langle f(x)\frac{\partial}{\partial \tau} h(\tau,x), \psi_n(x) \Big\rangle \Big)^2 d\tau \Big\} \\
&\quad \le C_9 \Big\{ \|f(x)h(0,x)\|^2_{L_2(\Omega)} + \frac{T^{1-\beta}}{r^\beta(1-\beta)} \left\| f(x)\frac{\partial}{\partial \tau} h(\tau,x) \right\|^2_{L_2(\Omega_T)} \Big\} \\
&\quad \le C_{10} \|f(x)\|^2_{L_2(\Omega)}.
\end{aligned}
$$

Since $H_0^\alpha(\Omega)$ is compactly embedded in $L^2(\Omega)$, see [1] and [18], the operator $h(T,x)A_r f$ is compact. Finally by using (2.1), we conclude that the operator $A_r : L^2(\Omega) \to L^2(\Omega)$ is compact. This completes the proof. □

Theorem 3.2. *$A_r f : I \to L^2(\Omega)$, defined by (2.16), is real analytic in $r \in I$ for arbitrarily fixed $f \in L^2(\Omega)$.*

Proof. We use the last theorem on page 65 of [7]. To show the real analyticity in $r \in I$ for $u(r,f) := u(r)$, it is enough to prove the following

(a) $\dfrac{\partial^\beta}{\partial t^\beta}u(r) \in C^\infty\Big(0,\infty; L_2(\Omega)\Big)$,

(b) $\left\|\dfrac{d^m}{dr^m}\Big(\dfrac{\partial^\beta}{\partial r^\beta}u(r)(T,x)\Big)\right\|_{L_2(\Omega)} \le M\ \eta^{-m}\ m!$, for $m \in \mathbb{N}$, $r \in J := [r_0, R_0] \subset I$, $M > 0$ and $\eta > 0$.

By (3.4) and the compact embedding, we have

$$\left\|\frac{\partial^\beta}{\partial t^\beta}u(r)(T,x)\right\|_{L_2(\Omega)} \le M.$$

By (3.3) and $\dfrac{d^m}{dr^m}E_\beta\big(-\lambda r^\beta\tau^\beta\big) = -\lambda_n r^{\beta-m}\tau^\beta E_{\beta,\beta-m+1}\big(-\lambda r^\beta\tau^\beta\big)$ we have

(3.5)
$$\frac{d^m}{dr^m}\Big(\frac{\partial^\beta}{\partial t^\beta}u(T,x)\Big) = \sum_{n=1}^{\infty}\Big(\Big\langle f(x)h(0,x),\psi_n(x)\Big\rangle(-\lambda_n)r^{\beta-m}T^\beta E_{\beta,\beta-m+1}\big(-\lambda_n r^\beta T^\beta\big)$$
$$-\int_0^T\Big\langle f(x)\frac{\partial}{\partial\tau}h(T-\tau,x),\psi_n(x)\Big\rangle(-\lambda_n)r^{\beta-m}\tau^\beta E_{\beta,\beta-m+1}\big(-\lambda_n r^\beta\tau^\beta\big)d\tau\Big)\psi_n(x).$$

Therefore we obtain

(3.6)
$$\left\|\frac{d^m}{dr^m}\Big(\frac{\partial^\beta}{\partial t^\beta}u(T,x)\Big)\right\|^2_{L_2(\Omega)} \le 2\sum_{n=1}^{\infty}\Big\{\Big\langle f(x)h(0,x),\psi_n(x)\Big\rangle^2\lambda_n^2 r^{2\beta-2m}T^\beta\Big(\frac{C_{11}}{1+\lambda_n r^\beta T^\beta}\Big)^2$$
$$+\Big(\int_0^T\Big\langle f(x)\frac{\partial}{\partial\tau}h(T-\tau,x),\psi_n(x)\Big\rangle^2\lambda_n^2 r^{2\beta-2m}d\tau\Big)\Big(\int_0^T\tau^{2\beta}\Big(\frac{C_{11}}{1+\lambda_n r^\beta T^\beta}\Big)^2 d\tau\Big)\Big\}$$
$$\le \frac{2C_{11}^2}{r^{2m}}\Big\{\big\|f(x)h(0,x)\big\|^2_{L_2(\Omega)} + T\big\|f\frac{\partial}{\partial\tau}h\big\|^2_{L_2((0,T)\times\Omega)}\Big\} \le M^2 r^{-2m},$$

where $M^2 = 2C_{11}^2\Big\{\big\|f(x)h(0,x)\big\|^2_{L_2(\Omega)} + T\big\|f\dfrac{\partial}{\partial\tau}h\big\|^2_{L_2((0,T)\times\Omega)}\Big\}$. Finally, by (2.1) and (2.16) we conclude that

$$\left\|\frac{d^m}{dr^m}A_r f\right\|_{L_2(\Omega)} \le M\delta^{-1}r_0^{-m}.$$

The proof is completed. □

Theorem 3.3. *There exists a constant* $0 < \mathcal{C}(r) < 1$ *such that*

$$\|A_r f\|_{L^2(\Omega)} \le \mathcal{C}(r)\|f\|_{L^2(\Omega)}, \tag{3.7}$$

where $R^\star < r$ and $R^\star > 0$ is a large number.

Proof. By (2.1), Theorem 2.1 and (3.3) we have

$$\left\|A_r(f)\right\|^2_{L_2(\Omega)} \le \frac{2}{\delta^2}\sum_{n=1}^{\infty}\Bigg(\underbrace{\left(\frac{C_{11}}{1+\lambda_n r^\beta T^\beta}\right)^2\left\langle f(x)h(0,x),\psi_n(x)\right\rangle^2}_{=:I_1} + \underbrace{\left(\int_0^T\left\langle f(x)\frac{\partial}{\partial\tau}h(T-\tau,x),\psi_n(x)\right\rangle E_\beta\big(-\lambda_n r^\beta\tau^\beta\big)d\tau\right)^2}_{=:I_2}\Bigg) \tag{3.8}$$

Now we estimate I_1 and I_2. For I_1 we have

$$I_1 \le \frac{C_{11}^2}{\lambda_1^2 r^{2\beta}T^{2\beta}}\left\langle f(x)h(0,x),\psi_n(x)\right\rangle^2. \tag{3.9}$$

For I_2 we have

$$\begin{aligned} I_2 &\le \int_0^T\left\langle f(x)\frac{\partial}{\partial\tau}h(T-\tau,x),\psi_n(x)\right\rangle^2 d\tau\int_0^T\left(E_\beta\big(-\lambda_n r^\beta\tau^\beta\big)\right)^2 d\tau \\ &\le \frac{C_{11}^2}{\lambda_1 r^\beta}\frac{T^{1-\beta}}{1-\beta}\int_0^T\left\langle f(x)\frac{\partial}{\partial\tau}h(T-\tau,x),\psi_n(x)\right\rangle^2 d\tau. \end{aligned} \tag{3.10}$$

If we substitute (3.9) and (3.10) in (3.8) and use Parseval's identity, we get

$$\begin{aligned} &\left\|A_r f\right\|^2_{L_2(\Omega)} \\ &\le \underbrace{\frac{C_{11}^2}{\delta^2}\left(\frac{1}{\lambda_1^2 r^{2\beta}T^{2\beta}}\|h(0,x)\|^2_{L_\infty(\Omega)} + \frac{1}{\lambda_1 r^\beta}\frac{T^{1-\beta}}{1-\beta}\|\partial_t h\|^2_{L_\infty((0,T)\times\Omega)}\right)}_{=:I_3}\|f\|^2_{L_2(\Omega)}. \end{aligned} \tag{3.11}$$

For large $r > 0$, $I_3 < 1$. This completes the proof. □

The following corollary is obtained directly by Theorem 3.3.

Corollary 3.1. *1 is not an eigenvalue of the operator A_r for large $r > 0$.*

4. Proof of the main result

In the proof of the main result of the paper, we will use the Analytic Fredholm Theorem. For the sake of the completeness we recall it here, (see page 266, Theorem 8.92 in [14]).

Theorem 4.1. *(Analytic Fredholm Theorem) Let $G \subset \mathbb{C}$ be a domain and L be a bounded linear operator on G. If L is real analytic on G and the operator $L(\lambda)$ is compact for each $\lambda \in G$ then either $\big(I - L(\lambda)\big)^{-1}$ does not exist for any $\lambda \in G$ or $\big(I - L(\lambda)\big)^{-1}$ exists for every $\lambda \in G \setminus S$ where S is a discrete set in G.*

Now we are ready to state and prove the main result of the paper.

Theorem 4.2. *There exists a finite set $S \subset I$ such that for $r \in I \setminus S$ and $\varphi \in H_0^\alpha(\Omega)$, the inverse problem (1.1), (1.3) has a unique solution. Moreover there exists a constant $C_{12} > 0$ such that*

$$\big\|f\big\|_{L^2(\Omega)} + \big\|u\big\|_{L_2(0,T;H_0^\alpha(\Omega))} + \big\|\partial_t^\beta u\big\|_{L_2(0,T;L_2(\Omega))} \le C_{12}\big\|\varphi\big\|_{H_0^\alpha(\Omega)}. \tag{4.1}$$

Proof. If we use Corollary 3.1 and apply Theorem 3.4 to the operator A_r, we see that the first alternative can not occur that is, $\big(I - A_r\big)^{-1}$ exists for every $\lambda \in I \setminus S$ where S is a discrete set in I. By Lemma 2.1, we conclude that the inverse problem (1.1), (1.3) is uniquely solvable. By (2.10), (2.15) and Theorem 3.3, we also conclude

$$\big\|f\big\|_{L^2(\Omega)} \le C_{12}\big\|\varphi\big\|_{H_0^\alpha(\Omega)}. \tag{4.2}$$

(4.2) with (2.8) implies (4.1). The proof is completed. □

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