SIMULTANEOUS INVERSION FOR THE EXPONENTS OF THE FRACTIONAL TIME AND SPACE DERIVATIVES IN THE SPACE-TIME FRACTIONAL DIFFUSION EQUATION

SALİH TATAR, RAMAZAN TINAZTEPE, AND SÜLEYMAN ULUSOY

ABSTRACT. Fractional(nonlocal) diffusion equations replace the integer-order derivatives in space and time by their fractional-order analogues and they are used to model anomalous diffusion, especially in physics. This paper is devoted to a nonlocal inverse problem related to the space-time fractional equation $\frac{\partial^\beta}{\partial t^\beta}u(t,x) = -(-\Delta)^{\alpha/2}u(t,x),\ -1 < x < 1,\ 0 < t < T$. The existence of the solution for the inverse problem is proved by using quasi-solution method which is based on minimizing an error functional between the output data and the additional data. In this context, an input-output mapping is defined and continuity of the mapping is established. The uniqueness of the solution for the inverse problem is also proved by using eigenfunction expansion of the solution and some basic properties of fractional Laplacian. A numerical method based on discretization of the minimization problem, steepest descent method and least squares approach is proposed for the solution of the inverse problem. The numerical method determines the exponents of the fractional time and space derivatives simultaneously. Numerical examples with noise free and noisy data illustrate applicability and high accuracy of the proposed method.

Keywords. Fractional derivative; Fractional Laplacian; Inverse problem; Existence and uniqueness; Minimization problem; Input-output map; Steepest descent method; Least squares approach

AMS Subject Classifications. 34A12, 35R11, 35R30, 65F22.

1. INTRODUCTION

In this paper, we study an inverse problem associated with the following one dimensional space-time fractional diffusion problem

$$(1.1)\quad \begin{cases} \frac{\partial^\beta}{\partial t^\beta}u(t,x) = -(-\Delta)^{\alpha/2}u(t,x),\ -1 < x < 1,\ 0 < t < T, \\ u(t,-1) = u(t,1) = 0,\ 0 < t < T, \\ u(0,x) = f(x),\ -1 < x < 1, \end{cases}$$

where $T > 0$ is a final time, $f(x) \in L^2(-1,1)$ is an initial function, $\frac{\partial^\beta}{\partial t^\beta}$ is the Caputo fractional time derivative, $(-\Delta)^{\alpha/2}$ is the fractional Laplacian,

$\beta \in \left(0, \frac{1}{2}\right)$ and $\alpha \in \left(\frac{1}{2}, 2\right)$ are fractional order of the time and the space derivatives respectively. For given inputs β, α and $f(x)$, the problem (1.1) is called the direct (forward) problem. Like most direct problems of the mathematical physics, the problem (1.1) is well-posed, see the very recent interesting paper[14] for details.

The inverse problem here consists of simultaneous determining the exponents β and α of the fractional time and space derivatives, by means of the observation data $u(t, 0) = g(t), 0 < t < T$. By this result one can expect that by means of experiments the important parameters β and α characterizing the anomalous diffusion can be identified simultaneously.

The classical diffusion equation $\partial_t u = \Delta u$ is used to describe a cloud of spreading particles at the macroscopic level. The point source solution is a Gaussian probability density that predicts the relative particle concentration. For microscopic picture, Brownian motion is employed, which describes the path of individual particles. The space-time fractional diffusion equation $\partial_t^\beta u = -(-\Delta)^{\alpha/2} u$ with $0 < \beta < 1$ and $0 < \alpha < 2$ is used to model anomalous diffusion [17]. Here, the fractional derivative in time is used to describe particle sticking and trapping phenomena and the fractional space derivative is used to model long particle jumps. These two effects combined together produces a concentration profile with a sharper peak, and heavier tails.

In fractional diffusion equations the fractional time derivative with $0 < \beta < 1$ is used to model slow diffusion, and the exponent β is related to the parameter specifying the large-time behavior of the waiting-time distribution function, see [20] and some of the references cited therein.

Recently, there has been a growing interest in inverse problems with fractional derivatives. Usually, in these works a fractional time derivative is considered and determination of that under some additional condition(s) is the inverse problem. These problems are physically and practically very important. We list some of the important references [3, 6, 9, 16, 22, 23, 26, 27, 28]. The difference of the current study from these works is that there are two different parameters, which are the orders of time and space fractional derivatives to be determined in the inverse problem considered. Furthermore, we determine these parameters simultaneously. This is a very recent approach in the inverse problems community, see [29] and some of the references cited therein. On the other hand, in this paper, we follow a similar approach and so the proofs are based on the eigenfunction expansion of the weak solution to the initial/boundary value problem. This study can be regarded as continuation of the series of works mentioned above on fractional inverse problems.

This paper is organized as follows: In the next section, for the sake of the reader, we remind the analysis of the direct problem and introduce the inverse problem. Section 3 includes both the statement and the proof of the existence and uniqueness theorem. In the fourth section the inversion algorithm is established theoretically. This algorithm is tested on two examples with noisy and noise free additional data in Section 5. The conclusions and possible directions on the problem are given in Section 6.

2. The direct and the inverse problems

As mentioned in Section 1, the fractional-time derivative considered in (1.1) is the Caputo fractional derivative of order $0 < \beta < 1$ and is defined by

$$\frac{\partial^\beta f(t)}{\partial t^\beta} := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial f(r)}{\partial r} \frac{dr}{(t-r)^\beta}, \tag{2.1}$$

where Γ is the Gamma function. This was intended to properly handle initial values [1, 2, 4], since its Laplace transform(LT) $s^\beta \tilde{f}(s) - s^{\beta-1} f(0)$ incorporates the initial value in the same way as the first derivative. Here, $\tilde{f}(s)$ is the usual Laplace transform. It is well-known that the Caputo derivative has a continuous spectrum [2], with eigenfunctions given in terms of the Mittag-Leffler function

$$E_\beta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

In fact, it is easy to see that, $f(t) = E_\beta(-\lambda t^\beta)$ solves the eigenvalue problem

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = -\lambda f(t), \quad f(0) = 1,$$

for any $\lambda > 0$. This is easily verified by differentiating term-by-term and using the fact that t^p has Caputo derivative $t^{p-\beta} \dfrac{\Gamma(p+1)}{\Gamma(p+1-\beta)}$ for $p > 0$ and $0 < \beta \leq 1$. See [2] for details.

If f is C^1-function on $[0,\infty)$ satisfying $|f'(t)| \leq Ct^{\gamma-1}$ for some $\gamma > 0$, then by (2.1), the Caputo derivative $\dfrac{\partial^\beta f(t)}{\partial t^\beta}$ of f exists for all $t > 0$ and the derivative is continuous in $t > 0$. Kilbas et al [13] and Podlubny [20] can be referred for further properties of the Caputo derivative.

For $0 < \alpha < 2$, in (1.1) $(-\Delta)^{\alpha/2} u$ denotes the fractional Laplacian. It turns out that it is easier to define it by using the spectral decomposition of the Laplace operator: We take $\{\bar{\lambda}_k, \psi_k\}$ the eigenvalues and corresponding eigenvectors of the Laplacian operator in $\Omega := (-1,1)$ with Dirichlet boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta\psi_k = \bar{\lambda}_k\psi_k, & \text{in} \quad \Omega, \\ \psi_k = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

We then define the operator $-(-\Delta)^{\alpha/2}$ by

$$-(-\Delta)^{\alpha/2}u := \sum_{k=0}^{\infty} c_k\psi_k(x) \mapsto -\sum_{k=0}^{\infty} c_k\bar{\lambda}_k^{\alpha/2}\psi_k(x),$$

which maps $H^\alpha(\Omega)$ onto $L^2(\Omega)$.

The equation

$$\frac{\partial^\beta}{\partial t^\beta}u(t,x) = -(-\Delta)^{\alpha/2}u(t,x),$$

is the standard linear evolution equation involving fractional diffusion. This is a model of so-called anomalous diffusion, a much studied topic in physics, probability and finance. There are many studies related to the direct problem for the equation

$$\frac{\partial^\beta}{\partial t^\beta}u(t,x) = (Lu)(t,x) + F(t,x),$$

in the literature (see [10], [11], [23] and some of the references cited therein), where L is symmetric uniformly elliptic operator. In these studies, the direct problems are formulated under different boundary conditions and a formula is derived for the solution using eigenfunction expansion. But only a few of them involve fractional Laplace operator in spite of its physical and practical importance (see [2], [12], [25] for example). The fractional powers of the classical Laplace operator, namely $-(-\Delta)^{\alpha/2}$ are particular cases of the infinitesimal generators of Lévy stable diffusion processes and appear in anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics (see [12] and [18] for an extensive list of current applications).

First of all, we need to define a solution formula for the direct problem (1.1). By using eigenfunction expansion method, following [2], [10], [23], [25], we get the following useful formula for the weak solution of the direct problem (1.1)

$$u(t,x) = \sum_{n=1}^{\infty} < f, \psi_n > E_\beta(-\lambda_n t^\beta)\psi_n(x), \tag{2.2}$$

the series is convergent in $C((0,T]; H^\alpha(-1,1))$ where $\lambda_n = \left(\bar{\lambda}_n\right)^{\alpha/2}$, $\bar{\lambda}_n$ and $\{\psi_n\}_{n\geq 1}$ are eigenvalues and eigenvectors of the classical Laplace operator Δ respectively, i.e, $-\Delta\psi_n = \bar{\lambda}_n\psi_n$. A simple calculation yields $\bar{\lambda}_n = \dfrac{n^2\pi^2}{4}$, $n \geq$

$1, n$ is odd so that $\lambda_n = (\frac{n\pi}{2})^{\alpha},\ n \geq 1, n$ is odd. $< \cdot\, , \cdot >$ denotes the standard inner product on $L^2(-1,1)$, $E_\beta(z) = E_{\beta,1}(z)$ and $E_{\beta,\alpha}(z)$ is the generalized Mittag-Leffler function defined as follows

$$E_{\beta,\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha + \beta k)}.$$

We note that $\{\lambda_n\}_{n\geq 1}$ is a sequence of positive numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\{\psi_n\}_{n\geq 1}$ is an orthonormal basis for $L^2(-1,1)$ and any $f \in L^2(-1,1)$ has the representation

$$f(x) = \sum_{n=1}^{\infty} < f, \psi_n > \psi_n(x).$$

In [2], the authors prove that (2.2) is also a strong solution of the direct problem (1.1) using the regularity property of $u(t,x)$. The series on the right-hand side of (2.2) is uniformly convergent in $x \in [-1,1]$ and $t \in (0,T]$, see [23] and [25] for details.

In the existence and uniqueness theorem, we will need the solution of the problem in the following form

$$\begin{cases} \dfrac{\partial^\beta}{\partial t^\beta} u(t) = -(-\Delta)^{\alpha/2} u(t) + h(t),\ t > 0, \\ u(0) = g. \end{cases} \tag{2.3}$$

For this purpose, we set

$$U(t)g = \sum_{n=1}^{\infty} \langle g, \psi_n \rangle E_\beta(-\lambda_n t^\beta) \psi_n(x),\ t \geq 0,$$

and

$$V(t)g = t^{\beta-1} \sum_{n=1}^{\infty} \Big(\langle g, \psi_n \rangle E_{\beta,\beta}(-\lambda_n t^\beta) \Big) \psi_n(x),\ t \geq 0,$$

where λ_n and $\psi_n(x)$ are eigenvalues and eigenfunctions of the operator $-A$ respectively. Then the solution of (2.3) is given by the following formula, see [10]

$$u(t) = U(t)g + \int_0^t V(t-s)h(s)ds,\ t > 0. \tag{2.4}$$

The following three theorems indicate some important properties of the Mittag-Leffler function, (see Theorem 1.4 and Theorem 1.6 in [20] respectively) which provide technical convenience in ensuing theorems ahead.

Lemma 2.1. *If $\beta < 2$, μ is such that $\frac{\pi\beta}{2} < \mu < \min\{\pi\beta, \pi\}$, $\mu \leq |\arg(z)| \leq \pi$, then the following expansion holds*

$$E_\beta(-z) = \frac{1}{z\Gamma(1-\beta)} + O(|z|^{-2}). \tag{2.5}$$

Lemma 2.2. *If $\beta < 2$, α is arbitrary real number, μ is such that $\frac{\pi\beta}{2} < \mu < \min\{\pi\beta, \pi\}$, $\mu \leq |\arg(z)| \leq \pi$, and C_0 is a real constant, then*

$$|E_{\beta,\alpha}(z)| \leq \frac{C_0}{1+|z|}. \tag{2.6}$$

Lemma 2.3. *If $0 \leq \beta \leq 1$, then $E_\beta(-z)$ is completely monotone on $(0,\infty)$ and all derivatives of $E_\beta(-z)$ are bounded on $(0,\infty)$.*

Proof. See [21] for the proof of the complete monotonicity of $E_\beta(-z)$. In the proof therein, we deduce $E_\beta(-z) > 0$ for $0 \leq \beta \leq 1$. Since it is complete monotone, $(-1)^n \frac{d^n}{dz^n} E_\beta(-z) \geq 0$, that is, $E_\beta^{(n)}(-z) \geq 0$ holds for each positive integer n. With the preceding fact and $\frac{d}{dz}\left(E_\beta^{(n)}(-z)\right) = -E_\beta^{(n+1)}(-z)$, we deduce $E_\beta^{(n)}(-z)$ is positive and decreasing hence bounded on $(0,\infty)$ for each positive integer n. □

Before dealing with the inverse problem, we first prove the regularity of the solution of the direct problem.

Theorem 2.1. *Let $f \in L^2(\Omega)$. There exists a unique weak solution $u \in C((0,T]; L^2(\Omega)) \cap C([0,T]; H^\alpha(\Omega))$ to (1.1) such that $\partial_t^\beta u \in C((0,T]; L^2(\Omega))$. Moreover, there exists a constant $C > 0$ such that*

$$||u(t,\cdot)||_{H^\alpha(\Omega)} + ||\partial_t^\beta u(t,\cdot)|| \leq C t^{-\beta} ||f||. \tag{2.7}$$

Proof. Existence and uniqueness of weak solution to (1.1) can be shown following [10, 23] . So we only prove (2.7). First,

$$||u(t,\cdot)||^2 = \left\| \sum_{n=1}^\infty < f, \psi_n > E_\beta(-\lambda_n t^\beta) \right\|^2 \leq \sum_{n=1}^\infty C^2 < f, \psi_n > \leq C||f||^2. \tag{2.8}$$

Following exactly the proof of (2.18) in [23] we have

$$||(-\Delta)^{\alpha/2} u|| \leq C t^{-\beta} ||f||. \tag{2.9}$$

In (2.8), since $\sum_{n=1}^\infty < f, \psi_n > E_\beta(-\lambda_n t^\beta)\psi_n$ is convergent in $L^2(\Omega)$ uniformly in $t \in [0.T]$, we see that $u \in C([0,T]; L^2(\Omega))$. Moreover, in (2.9), since $\sum_{n=1}^\infty \lambda_n < f, \psi_n > E_\beta(-\lambda_n t^\beta)\psi_n$ is convergent in $L^2(\Omega)$ uniformly in $t \in [\delta, T]$ with any given $\delta > 0$, we see that $-(-\Delta)^{\alpha/2} u \in C((0,T]; L^2(\Omega))$ that is $u \in C((0,T]; H^\alpha(\Omega))$. By (1.1) we see that $\partial_t^\beta u \in C((0,T]; L^2(\Omega))$ and consequently we have (2.7).

□

Next we define the inverse problem. As it is known, a direct problem aims to find a solution that satisfies given differential equation (ordinary, partial, or fractional) and related to initial and boundary conditions. In some problems, the main equation and the conditions are not sufficient to obtain the solution, but, instead some additional conditions (also called measured output data) are required. Such problems are called the inverse problems. In general, the additional conditions may be given on the boundary, on the final time or on the whole domain (also known as nonlocal condition). In this paper, we use the following additional condition of Dirichlet type

$$u(t,0) = g(t), \quad 0 < t < T. \tag{2.10}$$

The inverse problem here consists of determining the unknown orders β and α of the time and space derivatives in the space-time fractional diffusion problem (1.1) from the additional condition (2.10), that is, the problem (1.1),(2.10) is called the inverse problem for the given inputs $f(x)$ and $g(t)$.

In the existence and uniqueness theorem, due to some technical reasons in the proof of determining the exponents β and α, we will need a specific class of the initial functions $f(x)$ satisfying

$$\big\langle f(x), \psi_n(x) \big\rangle > 0 \quad \Big\{ \text{ or } \big\langle f(x), \psi_n(x) \big\rangle < 0 \Big\} \text{ for odd } n \geq 1, \tag{2.11}$$

where $\{\psi_n(x)\}_{n\geq 1}$ is an orthonormal basis for $L^2(-1,1)$. A similar class of functions is used to prove the uniqueness in [29], see also [25]. Throughout this paper, we assume that $g(t) \not\equiv 0$.

To the best of the authors' knowledge, there are not many works related to inverse problems for the fractional diffusion equations involving fractional Laplacian, see [25]. See also [15], where the author points out some open problems related to inverse problems involving fractional derivatives. Our current paper makes some contribution to this subject. Next section is devoted to the statement and the proof of the existence and uniqueness theorem for the inverse problem.

3. Statement and the proof of the main result

In this section, we state and prove the existence and uniqueness theorem. First we prove an existence theorem for a solution of the inverse problem. There are two main methods in the literature to prove existence of the solution of inverse problems for the classical diffusion equations. The first method is called the monotonicity method, which is based on the continuity and the monotonicity of the input-output mapping [5], [24]. The second method is called quasi-solution method, which is based on minimizing an error functional between the output data and the additional data [8], [19]. In this paper, we extend the quasi-solution method to the inverse problem for the space-time fractional diffusion equation. For this purpose, let $(\beta, \alpha) \in$

$[\beta_0,\beta_1]\times[\alpha_0,\alpha_1]$, where $\alpha>\frac{1}{2}$, $\beta,\hat\beta\in\left(0,\frac{1}{2}\right)$, $0<\beta_0\le\beta,\ \hat\beta\le\beta_1<\frac{1}{2}$, and denote a unique solution of the direct problem that corresponds to (β,α) by $u(\beta,\alpha)(x,t)$. We can obtain the output data $u(\beta,\alpha)(t,0)$ for $(\beta,\alpha)\in[\beta_0,\beta_1]\times[\alpha_0,\alpha_1]$ by using the formula (2.2). We use the notation $u(\beta,\alpha)(t,x)$ instead of $u(t,x)$ to emphasize the dependency of the solution to both β and α. An optimal idea for solving the inverse problem is to minimize an error functional between the output data and the additional data. For a given target function $\varphi\in L^2(0,T)$, the square integrable functions on $(0,T)$, we define the following minimization problem

$$\min_{(\beta,\alpha)\in[\beta_0,\beta_1]\times[\alpha_0,\alpha_1]}\left\|u(\beta,\alpha)(t,0)-\varphi\right\|, \tag{3.1}$$

throughout the paper we denote by $\|\cdot\|$ the norm in $L^2(\Omega)$, while $\|\cdot\|_X$ denotes the norm in the space X. We set the following input-output mapping

$$F(\beta,\alpha)(t):(\beta,\alpha)\longrightarrow u(\beta,\alpha)(t,0),\ 0<t<T. \tag{3.2}$$

where $F:[\beta_0,\beta_1]\times[\alpha_0,\alpha_1]\longrightarrow L^2(0,T)$. We now provide the well-posedness of this mapping: By (2.2), (2.10) and (2.6)

$$\begin{aligned}\int_0^T|u(t,0)|^2\,dt=\int_0^T|g(t)|^2\,dt&=\int_0^T|\sum_{n\ge1}<f,\psi_n>E_\beta(-\lambda_n t^\beta)\psi_n(0)|^2\,dt\\&\le\left[\frac{C}{\lambda_1^2}\sum_{n=1}^{\infty}<f,\psi_n>^2\psi_n(0)^2\right]\int_0^T\frac{1}{t^{2\beta}}\,dt.\end{aligned}$$

The last integral is convergent provided that $\beta<\frac{1}{2}$. Now, we prove the following theorem for the input output mapping (3.2).

Theorem 3.1. *The input-output mapping, defined by* (3.2) *is continuous.*

Proof. We regard $u(t,x)$ as a mapping from $t\in(0,T)$ to $L^2(-1,1)$ and write $u(t)=u(t,\cdot)$. Let $u=u(\beta,\alpha),v=u(\hat\beta,\hat\alpha),y=u-v,\ \hat\beta>\beta$. We see that y solves the following problem

$$\begin{cases}\dfrac{\partial^\beta}{\partial t^\beta}y=-(-\Delta)^{\alpha/2}y\underbrace{+(-\Delta)^{\hat\alpha/2}v-(-\Delta)^{\alpha/2}v}_{=:I_1}\underbrace{-\dfrac{\partial^\beta}{\partial t^\beta}v+\dfrac{\partial^{\hat\beta}}{\partial t^{\hat\beta}}v}_{=:I_2},\\[2ex] y(0)=0.\end{cases} \tag{3.3}$$

We now estimate I_1 and I_2. To estimate I_1 we note that by the definition of the fractional Laplacian and the Mean Value Theorem

$$
\begin{aligned}
I_1 &= \sum_{k=0}^{\infty} c_k[\bar{\lambda}_k^{\hat{\alpha}/2} - \bar{\lambda}_k^{\hat{\alpha}/2}]\psi_k(x) \\
&\le C\sum_{k=0}^{\infty} c_k|\alpha-\hat{\alpha}|\bar{\lambda}^{\tilde{\gamma}}\psi_k(x), \\
&\le C|\alpha-\hat{\alpha}|\sum_{k=0}^{\infty} c_k\bar{\lambda}_k^{\tilde{\gamma}}\psi_k(x) \\
&= C|\alpha-\hat{\alpha}|(-\Delta)^{\tilde{\gamma}}v
\end{aligned} \tag{3.4}
$$

where $\tilde{\gamma}$ is a number between $\dfrac{\alpha}{2}$ and $\dfrac{\hat{\alpha}}{2}$. So, using the estimate (2.7)

$$
\begin{aligned}
||I_1(t)|| &\le C|\alpha-\hat{\alpha}|\ ||(-\Delta)^{\tilde{\gamma}}v|| \\
&\le C|\alpha-\tilde{\alpha}|\, t^{-\tilde{\gamma}}\, ||f||.
\end{aligned} \tag{3.5}
$$

For I_2 the following estimate holds

$$
\left\|I_2\right\| \le C\left|\beta-\hat{\beta}\right|(1+t^{-\hat{\beta}}). \tag{3.6}
$$

We sketch the proof of (3.6), which follows [29] closely. Note that

$$
\begin{aligned}
I_2 &= \left[1-\frac{\Gamma(1-\hat{\beta})}{\Gamma(1-\beta)}\right]\frac{1}{\Gamma(1-\hat{\beta})}\int_0^t (t-s)^{-\hat{\beta}}v'(s)\,ds \\
&\quad -\frac{1}{\Gamma(1-\beta)}\int_0^t\left[(t-s)^{-\beta}-(t-s)^{-\hat{\beta}}\right]v'(s)\,ds \\
&=: I_{2,1}+I_{2,2}.
\end{aligned} \tag{3.7}
$$

First, by $0<\beta_0\le\beta,\hat{\beta}\le\beta_1<1$

$$
\begin{aligned}
||I_{2,1}(t)|| &\le C|\Gamma(1-\hat{\beta})-\Gamma(1-\beta)|\ ||\partial_t^{\hat{\beta}}v|| \\
&\le C|\beta-\hat{\beta}|t^{-\hat{\beta}}, \quad 0<t<T,
\end{aligned} \tag{3.8}
$$

where we have used the regularity estimate of the forward problem (2.8), $f\in L^2(\Omega)$ and the Lipschitz continuity of the Gamma function. Now, by definition

$$
||v'|| = Ct^{\hat{\beta}-1}\left\|\sum_{n=1}^{\infty}\lambda_n < f,\psi_n > E_{\hat{\beta}}(-\lambda_n t^{\hat{\beta}})\psi_n\right\|.
$$

Hence, since $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$, we see that

$$
||v'(t)|| \le Ct^{\hat{\beta}-1}, \quad 0<t<T. \tag{3.9}
$$

Thus, by (3.9) we have

$$
\begin{aligned}
||I_{2,2}(t)|| &\leq C\int_0^t |(t-s)^{-\beta}-(t-s)^{-\hat{\beta}}|\,||v'(s)||\,ds \\
&\leq C\int_0^t |(t-s)^{-\beta}-(t-s)^{-\hat{\beta}}|s^{\hat{\beta}-1}\,ds.
\end{aligned}
\tag{3.10}
$$

The estimation of this last term (3.10) follows exactly from [23]. Hence we have (3.6).

For sufficiently small and fixed $\epsilon>0$, we define a fractional power $\Big(-(-\Delta)^{\alpha/2}\Big)^{1+\frac{\epsilon}{\beta}}$ (for details, see [29] and references therein)

$$
\left\| \Big(-(-\Delta)^{\alpha/2}\Big)^{1+\frac{\epsilon}{\beta}} z\right\| = \left(\sum_{n=1}^{\infty}\lambda_n^{2+\frac{2\epsilon}{\beta}}\big\langle z,\psi_n\big\rangle^2\right)^{\frac{1}{2}} < \infty.
$$

If we apply (2.4) to (3.3) for $g=0$, we get

$$
\begin{aligned}
\big(-(-\Delta)^{\alpha/2}\big)^{\frac{1}{4}+\epsilon}y(\tau) = &\int_0^{\tau}\big(-(-\Delta)^{\alpha/2}\big)^{\frac{1}{4}+\epsilon}V(\tau-s) \\
&\times\left[-(-\Delta)^{\hat{\alpha}/2}v+(-\Delta)^{\alpha/2}v+\frac{\partial^{\beta}}{\partial t^{\beta}}v-\frac{\partial^{\hat{\beta}}}{\partial t^{\hat{\beta}}}v\right]ds,\ t>0.
\end{aligned}
\tag{3.11}
$$

By using (2.6) and Parseval identity, we have for some function $z\in L^2(\Omega)$

$$
\begin{aligned}
\left\|\big(-(-\Delta)^{\alpha/2}\big)^{\frac{1}{4}+\epsilon}V(\tau)z\right\| &= \left\|\sum_{n=1}^{\infty}\tau^{\beta-1}\big\langle z,\psi_n\big\rangle E_{\beta,\beta}(-\lambda_n\tau^{\beta})\lambda_n^{\frac{1}{4}+\epsilon}\psi_n\right\| \\
&\leq C_7\left(\sum_{n=1}^{\infty}\tau^{2\beta-2}\big\langle z,\psi_n\big\rangle^2\left(\frac{1}{1+\lambda_n\tau^{\beta}}\right)^2\lambda_n^{\frac{1}{2}+2\epsilon}\right)^{\frac{1}{2}} \\
&\leq C_7\left(\tau^{2\beta-2}\tau^{-(\frac{1}{2}+2\epsilon)\beta}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\big\langle z,\psi_n\big\rangle^2\underbrace{\left(\frac{\big(\lambda_n\tau^{\beta}\big)^{\frac{1}{4}+\epsilon}}{1+\lambda_n\tau^{\beta}}\right)^2}_{=:I_3}\right)^{\frac{1}{2}} \\
&\leq C_7\tau^{(\frac{3}{4}-\epsilon)\beta-1}\|z\|.
\end{aligned}
\tag{3.12}
$$

In the last step, we use that $I_3 < 1$. We conclude, by taking $z = I_1(s)+I_2(s)$ in (3.12), that

(3.13)
$$\begin{aligned}&\left\|\big(-(-\Delta)^{\alpha/2}\big)^{\frac{1}{4}+\epsilon}y(\tau)\right\|\\ &\leq C_7\int_0^\tau(\tau-s)^{(\frac{3}{4}-\epsilon)\beta-1}\left(\left\|-(-\Delta)^{\hat{\alpha}/2}v+(-\Delta)^{\alpha/2}v\right\|+\left\|\frac{\partial^\beta}{\partial t^\beta}v-\frac{\partial^{\hat{\beta}}}{\partial t^{\hat{\beta}}}v\right\|\right)ds.\\ &\leq C_8\int_0^\tau(\tau-s)^{(\frac{3}{4}-\epsilon)\beta-1}\left[s^{-\tilde{\gamma}}|\alpha-\hat{\alpha}|+\left|\beta-\hat{\beta}\right|\left(1+s^{-\hat{\beta}}\right)\right]ds.\end{aligned}$$

Employing the Sobolev embedding $H^{\frac{1}{2}+2\epsilon}$ in $C([0,1])$ and noting that the time integrals in (3.13) are convergent we complete the proof the Theorem 3.1. □

For practical use in latter sections we define the following functional

$$I(a) := \left\|u(a)(t,0)-\varphi\right\|_{L^2(0,T)}, \tag{3.14}$$

where $a = (\beta, \alpha) \in [\beta_0, \beta_1] \times [\alpha_0, \alpha_1]$. An application of the usual argument on the compactness of the interval $[\beta_0, \beta_1] \times [\alpha_0, \alpha_1] \subset \mathbb{R}^2$ yields the following existence theorem.

Theorem 3.2. *There exists $(\beta^\star, \alpha^\star) \in [\beta_0, \beta_1] \times [\alpha_0, \alpha_1]$ such that*

$$\left\|u(\beta^\star,\alpha^\star)(t,0)-\varphi\right\|_{L^2(0,T)} \leq \left\|u(\beta,\alpha)(t,0)-\varphi\right\|_{L^2(0,T)}.$$

The following theorem is particularly important in providing the uniqueness of the solution because it does not use the classical approaches such as using extra additional data or restricting the set of the admissible solutions. Instead, it requires only one additional data.

Theorem 3.3. *Assume that the condition (2.11) holds. Let u be the solution of the problem (1.1) and let v be the solution of the following equation with the same initial and boundary conditions:*

$$\frac{\partial^{\hat{\beta}}}{\partial t^{\hat{\beta}}}u(t,x) = -(-\Delta)^{\hat{\alpha}/2}u(t,x),\ -1<x<1,\ 0<t<T. \tag{3.15}$$

If $u(t,0) = v(t,0), 0 < t < T$, then $\beta = \hat{\beta}$ and $\alpha = \hat{\alpha}$.

Proof. By using the explicit formula (2.2), the solutions $u(t,x)$ and $v(t,x)$ can be written as follows

$$u(t,x) = \sum_{n=1}^{\infty} E_\beta(-\lambda_n t^\beta) < f, \psi_n > \psi_n(x), \tag{3.16}$$

and

$$v(t,x)=\sum_{n=1}^{\infty}E_{\hat{\beta}}(-\mu_n t^{\hat{\beta}}) < f,\psi_n > \psi_n(x). \tag{3.17}$$

Here $\lambda_n=\big(\bar{\lambda}_n\big)^{\alpha/2}$, $\mu_n=\big(\bar{\lambda}_n\big)^{\hat{\alpha}/2}$ where $\bar{\lambda}_n$ and $\{\psi_n\}_{n\geq 1}$ are eigenvalues and eigenvectors of the classical Laplace operator Δ respectively, i.e, $-\Delta\psi_n=\bar{\lambda}_n\psi_n$.

Consequently, assuming that $u(t,0)=v(t,0)$ we have

$$\sum_{\substack{n\geq 1\\ n \text{ is odd}}} E_{\beta}(-\lambda_n t^{\beta}) < f,\psi_n >= \sum_{\substack{n\geq 1\\ n \text{ is odd}}} E_{\hat{\beta}}(-\mu_n t^{\hat{\beta}}) < f,\psi_n >,\ 0<t\leq T. \tag{3.18}$$

Since both sides of (3.18) are analytic in Re $t>0$ we have

$$\sum_{\substack{n\geq 1\\ n \text{ is odd}}} E_{\beta}(-\lambda_n t^{\beta}) < f,\psi_n >= \sum_{\substack{n\geq 1\\ n \text{ is odd}}} E_{\hat{\beta}}(-\mu_n t^{\hat{\beta}}) < f,\psi_n >,\ t>0.$$

By using (2.5), there exists a constant $C_9>0$ such that the following inequality holds for large t

$$\Big|E_{\beta}(-\lambda_n t^{\beta})-\frac{1}{\Gamma(1-\beta)}\frac{1}{\lambda_n t^{\beta}}\Big| \leq \frac{C_9}{\lambda_n^2 t^{2\beta}}. \tag{3.19}$$

If we take the sum for odd integer $n\geq 1$ using the explicit form of λ_n we have

$$\sum_{\substack{n\geq 1\\ n \text{ is odd}}}\Big|E_{\beta}(-\lambda_n t^{\beta})-\frac{1}{\Gamma(1-\beta)}\frac{1}{\lambda_n t^{\beta}}\Big| \leq \frac{C_{10}}{t^{2\beta}}. \tag{3.20}$$

We add and subtract the term $\dfrac{1}{\Gamma(1-\beta)\lambda_n t^{\beta}}$ to the left hand side of (3.18) and use (3.20) to get the following asymptotic equality

$$\begin{aligned}&\sum_{\substack{n\geq 1\\ n \text{ is odd}}} E_{\beta}(-\lambda_n t^{\beta}) < f,\psi_n >\\ &=\sum_{\substack{n\geq 1\\ n \text{ is odd}}} < f,\psi_n > \frac{1}{\Gamma(1-\beta)}\frac{1}{\lambda_n t^{\beta}}+O\Big(\Big|\frac{1}{t^{2\beta}}\Big|\Big).\end{aligned} \tag{3.21}$$

Similarly, arguing for $\sum_{\substack{n\geq 1\\ n \text{ is odd}}} E_{\hat\beta}(-\mu_n t^{\hat\beta}) < f, \psi_n >$, we have

$$\sum_{\substack{n\geq 1\\ n \text{ is odd}}} E_{\hat\beta}(-\mu_n t^{\hat\beta}) < f, \psi_n > = \sum_{\substack{n\geq 1\\ n \text{ is odd}}} < f, \psi_n > \frac{1}{\Gamma(1-\hat\beta)} \frac{1}{\mu_n t^{\hat\beta}} + O\Big(\Big|\frac{1}{t^{2\hat\beta}}\Big|\Big). \tag{3.22}$$

Therefore from (3.18), (3.21) and (3.22) we have, as $t\to\infty$

$$\sum_{\substack{n\geq 1\\ n \text{ is odd}}} < f, \psi_n > \frac{1}{\Gamma(1-\beta)} \frac{1}{\lambda_n t^{\beta}} + O\Big(\Big|\frac{1}{t^{2\beta}}\Big|\Big) = \sum_{\substack{n\geq 1\\ n \text{ is odd}}} < f, \psi_n > \frac{1}{\Gamma(1-\hat\beta)} \frac{1}{\mu_n t^{\hat\beta}} + O\Big(\Big|\frac{1}{t^{2\hat\beta}}\Big|\Big). \tag{3.23}$$

For a moment, suppose that $\beta > \hat\beta$. Then multiplication of (3.23) by $t^{\hat\beta}$ yields that

$$-\frac{t^{\hat\beta}}{t^{\beta}} \sum_{\substack{n\geq 1\\ n \text{ is odd}}} < f, \psi_n > \frac{1}{\Gamma(1-\beta)} \frac{1}{\lambda_n} - O\Big(\Big|\frac{t^{\hat\beta}}{t^{2\beta}}\Big|\Big) + \sum_{\substack{n\geq 1\\ n \text{ is odd}}} < f, \psi_n > \frac{1}{\Gamma(1-\hat\beta)} \frac{1}{\mu_n} + O\Big(\Big|\frac{1}{t^{\hat\beta}}\Big|\Big) = 0. \tag{3.24}$$

Letting $t\to\infty$ in (3.24) , we deduce that

$$\sum_{\substack{n\geq 1\\ n \text{ is odd}}} < f, \psi_n > \frac{1}{\Gamma(1-\hat\beta)} \frac{1}{\mu_n} = 0. \tag{3.25}$$

Since the left-hand side of (3.25) is never zero, by (2.11) and positivity of the Gamma function on (0,1), we have a contradiction. Similarly, the assumption $\hat\beta > \beta$ leads to a contradiction. Therefore, we conclude that $\hat\beta = \beta$.

Now, we prove that $\alpha = \hat\alpha$. Without loss of generality, first assume that $\hat\alpha > \alpha$(a similar argument for the case $\alpha > \hat\alpha$ can be presented). By (3.18) and $\hat\beta = \beta$ we have

$$\sum_{\substack{n\geq 1\\ n \text{ is odd}}} E_{\beta}(-\lambda_n t^{\beta}) < f, \psi_n > = \sum_{\substack{n\geq 1\\ n \text{ is odd}}} E_{\beta}(-\mu_n t^{\beta}) < f, \psi_n > . \tag{3.26}$$

We take the Laplace transform of $E_\beta(-\lambda_n t^\beta)$ as follows:

$$\int_0^\infty e^{-zt} E_\beta(-\lambda_n t^\beta)\, dt = \frac{z^{\beta-1}}{z^\beta + \lambda_n}, \text{ Re } z > 0. \tag{3.27}$$

Moreover, if we take the Laplace transform of the Mittag-Leffler function term by term, we obtain

$$\int_0^\infty e^{-zt} E_\beta(-\lambda_n t^\beta)\, dt = \frac{z^{\beta-1}}{z^\beta + \lambda_n}, \text{ Re } z > \lambda_n^{\frac{1}{\beta}}. \tag{3.28}$$

Since $\sup_{t \geq 0} \left| E_\beta(-\lambda_n t^\beta) \right| < \infty$, by (2.6), we see that $\int_0^\infty e^{-zt} E_\beta(-\lambda_n t^\beta)\, dt$ is analytic in z for Re $z > 0$. Thus the analytic continuation yields (3.27) for Re $z > 0$. By using (2.6), and Lebesgue's convergence theorem, we get that $e^{-t\text{Re } z} t^{-\beta}$ is integrable for $t \in (0, \infty)$ with fixed z satisfying Re $z > 0$ and

$$\begin{aligned}
&\left| e^{-t\text{Re } z} \sum_{\substack{n \geq 1 \\ n \text{ is odd}}} < f, \psi_n > E_\beta(-\lambda_n t^\beta) \right| \\
&\leq C_0 e^{-t\text{Re } z} \left(\sum_{\substack{n \geq 1 \\ n \text{ is odd}}} < f, \psi_n > \frac{1}{|\lambda_n|} \frac{1}{t^\beta} \right) \\
&\leq \frac{C_0}{\pi^2} \frac{1}{t^\beta} \|f\| \; e^{-t\text{Re } z} \sum_{n=1}^\infty \frac{1}{n^2}.
\end{aligned}$$

Then, for Re $z > 0$ we obtain

$$\begin{aligned}
&\int_0^\infty e^{-zt} \sum_{\substack{n \geq 1 \\ n \text{ is odd}}} < f, \psi_n > E_\beta(-\lambda_n t^\beta)\, dt \\
&\qquad = \sum_{\substack{n \geq 1 \\ n \text{ is odd}}} < f, \psi_n > \frac{z^{\beta-1}}{z^\beta + \lambda_n}.
\end{aligned} \tag{3.29}$$

Similarly,

$$\begin{aligned}
&\int_0^\infty e^{-zt} \sum_{\substack{n \geq 1 \\ n \text{ is odd}}} < f, \psi_n > E_\beta(-\mu_n t^\beta)\, dt \\
&\qquad = \sum_{\substack{n \geq 1 \\ n \text{ is odd}}} < f, \psi_n > \frac{z^{\beta-1}}{z^\beta + \mu_n}.
\end{aligned} \tag{3.30}$$

Then, from (3.26), (3.29) and (3.30) we deduce that

$$\sum_{\substack{n \geq 1 \\ n \text{ is odd}}} \frac{< f, \psi_n >}{z^\beta + \lambda_n} = \sum_{\substack{n \geq 1 \\ n \text{ is odd}}} \frac{< f, \psi_n >}{z^\beta + \mu_n}, \text{ Re } z > 0, \tag{3.31}$$

or equivalently,

$$\sum_{\substack{n \geq 1 \\ n \text{ is odd}}} \frac{< f, \psi_n >}{\rho + \lambda_n} = \sum_{\substack{n \geq 1 \\ n \text{ is odd}}} \frac{< f, \psi_n >}{\rho + \mu_n}, \text{ Re } \rho > 0. \tag{3.32}$$

The equality (3.32) implies that

$$\sum_{\substack{n\geq 1\\ n \text{ is odd}}} \frac{< f,\psi_n > (\mu_n-\lambda_n)}{(\rho+\lambda_n)(\rho+\mu_n)} = 0,\ \text{Re}\ \rho > 0. \tag{3.33}$$

From (2.11), (3.33) and our assumption $\hat{\alpha} > \alpha$, we conclude that

$$\lambda_n = \mu_n,\ n \geq 1,\ n \ \text{ is odd.} \tag{3.34}$$

But, from (3.34) and $\lambda_n = \left(\bar{\lambda}_n\right)^{\alpha/2}$, $\mu_n = \left(\bar{\lambda}_n\right)^{\hat{\alpha}/2}$ we have a contradiction. Similarly, $\alpha > \hat{\alpha}$ leads to a contradiction. Therefore, we conclude that $\alpha = \hat{\alpha}$.

This completes the proof. □

In [25], the authors proved a uniqueness result in an inverse problem for a space-time fractional diffusion equation assuming $\psi_n(0) = 1$ which is only true for odd n. The proof there should be modified similar to this one.

4. The inversion algorithm

The inversion algorithm is based on the minimization of the error functional $I(a)$ defined by (3.14) where $a = (\beta, \alpha)$. We note that the continuity, hence the existence of the minimum of the functional on a compact set has been established in the previous section which is not enough to set up an efficient search algorithm for the minimum. Before developing an algorithm to find the minimum, we observe a key fact about the functional $I(a)$ which is differentiability. Now we prove that under certain conditions on f, $I(a)$ is differentiable with respect to a on a neighborhood of the minimum. This will enable us to implement a gradient method for the minimization.

Theorem 4.1. *The function $I(\beta,\alpha)$ is differentiable on $\left(\frac{1}{2}, 2\right) \times (0,1)$ if $|< f,\phi_n >| < \frac{1}{n^{1+2\alpha+\theta}}$ for some $\theta > 0$ and $\varphi(t)$ is bounded.*

Proof. Without loss of generality hereafter we take $T = 1$. First we note that

$$\begin{aligned} I(\beta,\alpha) &= \left\| u(\beta,\alpha)(t,0) - \varphi(t) \right\|^2_{L^2(0,1)} \\ &= \int_0^1 u(\beta,\alpha)^2(t,0)\,dt - 2\int_0^1 u(\beta,\alpha)(t,0)\varphi(t)\,dt + \int_0^1 \varphi^2(t)\,dt. \end{aligned}$$

To show the differentiability of $I(\beta,\alpha)$, first we need to show the differentiability of the integrands of the first two integrals for each t and continuity of partial derivatives. This is equivalent to showing the differentiability of $u(\beta,\alpha)(t,0)$ with respect to α and β for each t. Next the derivatives of the integrands with respect to α and β will be dominated by a function of t that is in $L^1[0,1]$ so that the result follows from the dominated convergence theorem. By the same reasoning, the continuity of the partial derivatives follows from the continuity of the derivatives of the integrands. Recall that an analytical expression of $u(\beta,\alpha)(t,x)$ is already present

$$u(\beta,\alpha)(t,x) = \sum_{\substack{n\geq 1 \\ n \text{ is odd}}}^{\infty} E_\beta(-\overline{\lambda}_n t^\beta) < f,\psi_n > \psi_n(x).$$

Since $\psi_n(0) = 1$ when n is even and zero when n is odd and $\overline{\lambda}_n = (\frac{n\pi}{2})^\alpha$, it is simplified to

$$u(\beta,\alpha)(t,0) = \sum_{\substack{n\geq 1 \\ n \text{ is odd}}}^{\infty} E_\beta(-(\frac{n\pi}{2})^\alpha t^\beta) < f,\psi_n > .$$

Now the partial derivative with respect to α is obtained as follows

$$\frac{\partial}{\partial\alpha}u(\beta,\alpha)(t,0) = \sum_{\substack{n\geq 1 \\ n \text{ is odd}}}^{\infty} \frac{\partial}{\partial\alpha}(E_\beta(-(\frac{n\pi}{2})^\alpha t^\beta)) < f,\psi_n >,$$

where

$$\frac{\partial}{\partial\alpha}(E_\beta(-(\frac{n\pi}{2})^\alpha t^\beta)) = -\frac{1}{2}\ln(\frac{n\pi}{2})t^\beta E'_\beta(-(\frac{n\pi}{2})^\alpha t^\beta).$$

Putting the expressions above together gives

$$\frac{\partial}{\partial\alpha}u(\beta,\alpha)(t,0) = -t^\beta \sum_{\substack{n\geq 1 \\ n \text{ is odd}}}^{\infty} \alpha\ln(\frac{n\pi}{2})E'_\beta(-\left(\frac{n\pi}{2}\right)^\alpha t^\beta) < f,\psi_n > . \tag{4.1}$$

We note that since $E_\beta(z)$ is an entire and completely monotone function, it is infinitely differentiable on the real line and $E'_\beta(-z)$ is bounded on $(0,\infty)$ by Theorem 2.3. With the fact that t^β is continuous on [0,1] and the assumption $| < f,\psi_n > | < \frac{1}{n^{1+2\alpha+\theta}}$ for some $\theta > 0$, we conclude there exists a positive number C such that the following inequality holds

$$\left| -t^\beta \sum_{\substack{n\geq 1 \\ n \text{ is odd}}}^{\infty} \frac{\alpha}{2}\ln(\frac{n\pi}{2})E'_\beta(-\left(\frac{n\pi}{2}\right)^\alpha t^\beta) < f,\psi_n > \right| < C \sum_{\substack{n\geq 1 \\ n \text{ is odd}}}^{\infty} \ln(\frac{n\pi}{2})\frac{1}{n^{1+\theta}} < \infty.$$

So $\frac{\partial}{\partial\alpha}u(\beta,\alpha)(t,0)$ exists for each t and bounded on $[0,1]$. Since $u(\beta,\alpha)(t,0)$ and ψ is bounded on [0,1], we conclude the differentiability of $I(\beta,\alpha)$ with respect to α. The continuity of the partial derivative of I with respect to α follows from the continuity of the expression (4.1) for each α and β.

The partial derivative with respect to β is obtained as follows

$$\frac{\partial}{\partial\beta}u(\beta,\alpha)(t,0) = \sum_{\substack{n\geq 1\\ n\text{ is odd}}}^{\infty} \frac{\partial}{\partial\beta}(E_\beta(-(\frac{n\pi}{2})^\alpha t^\beta)) < f,\psi_n >,$$

where

$$\frac{\partial}{\partial\beta}(E_\beta\left((-\frac{n\pi}{2})^\alpha t^\beta\right)) = \frac{\partial}{\partial\beta}\left(\sum_{k=0}^{\infty}\frac{\left(-(\frac{n\pi}{2})^\alpha t^\beta\right)^k}{\Gamma(1+\beta k)}\right)$$
$$= \sum_{k=0}^{\infty}\left(\frac{k\left((-\frac{n\pi}{2})^\alpha t^\beta\right)\ln t\left((-\frac{n\pi}{2})^\alpha t^\beta\right)^{k-1}}{\Gamma(1+\beta k)} - \frac{k\Gamma'(1+\beta k)\left((-\frac{n\pi}{2})^\alpha t^\beta\right)^k}{\Gamma(1+\beta k)^2}\right)$$
$$= \underbrace{\left((-\frac{n\pi}{2})^\alpha t^\beta\ln t\right)E'_\beta\left(-(\frac{n\pi}{2})^\alpha t^\beta\right)}_{:=A(t)} - \underbrace{\sum_{k=0}^{\infty}\frac{k\psi_0(1+\beta k)\left(-(\frac{n\pi}{2})^\alpha t^\beta\right)^k}{\Gamma(1+\beta k)}}_{:=B(t)},$$

where $\psi_0(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$, i.e, the digamma function. In the expression above $A(t)$ is defined for every $t \in (0,1]$ because $E_\beta(z)$ is differentiable everywhere and $B(t)$ is defined for every $t \in [0,1]$ because the radius of convergence is infinity. On the other hand, since $E'_\beta(-z)$ is bounded on $(0,\infty)$ by Theorem 2.3, $|A(t)| < M(\frac{n\pi}{2})^\alpha t^\beta|\ln t|$ for some $M > 0$. Since $\psi_0(1+\beta k) \approx \ln(1+\beta k)$ and $k\ln(1+\beta) \leq k\ln(1+\beta k) \leq k(k-1)$ for sufficiently large k , $B(t)$ remains between some multiples of $n^\alpha t^\beta E'_\beta(-(\frac{n\pi}{2})^\alpha t^\beta)$ and $n^{2\alpha}t^{2\beta}E''_\beta(-(\frac{n\pi}{2})^\alpha t^\beta)$ which means $B(t)$ is bounded, i.e., $|B(t)| < Nn^{2\alpha}$ for some $N > 0$ on $[0,1]$. Thus

$$\begin{aligned}
\left|\frac{\partial}{\partial\beta}u(\beta,\alpha)(t,0)\right| &< \sum_{\substack{n\geq 1\\ n\text{ is odd}}}^{\infty} \left|\frac{\partial}{\partial\beta}(E_\beta(-(\frac{n\pi}{2})^\alpha t^\beta))\right| |< f,\psi_n >| \\
&< \sum_{\substack{n\geq 1\\ n\text{ is odd}}}^{\infty} \frac{|A(t)|+|B(t)|}{n^{1+2\alpha+\theta}} \\
&< \sum_{\substack{n\geq 1\\ n\text{ is odd}}}^{\infty} \frac{\left(M\left(\frac{n\pi}{2}\right)^\alpha t^\beta|\ln t| + Nn^{2\alpha}\right)}{n^{1+2\alpha+\theta}} \\
&< C_1 t^\beta|\ln t| + C_2 \text{ for some } C_1 \text{ and } C_2 \\
&< C_1 t|\ln t| + C_2,
\end{aligned}$$

which is integrable on $[0,1]$. Now the boundedness of $u(\beta,\alpha)(t,0)$ and $\varphi(t)$ on [0,1] gives the differentiability. The continuity of the partial derivative with respect to β follows from the continuity of $A(t)$ and $B(t)$ with respect to β for each t. □

For the ill-posedness of the inverse problem, Tikhonov regularization is applied hence a regularization term with a regularization parameter λ is added to $I(a)$. Now we focus on minimizing the following function

$$F(a) = \left\| u(a)(t,0) - \varphi(t) \right\|^2_{L^2(0,1)} + \lambda \left\| a \right\|^2_E ,$$

where $\|a\|_E$ denotes the Euclidean norm of a.

We proceed the minimization of $F(a)$ by the steepest descent method which will utilize the gradient of F. In this method, the algorithm starts with an initial point b_0, then the point providing the minimum is approximated by the points

$$b_{i+1} = b_i + \triangle b_i,$$

where $\triangle b_i$ is the feasible direction which minimizes $F(b_i + \triangle b)$. This procedure is repeated until the algorithm gets sufficiently close to the minimum point. Since the minimum point is not known, one of the following stop criteria will be used: $\|\triangle b_i\|_E < \epsilon$ or $|F(b_{i+1}) - F(b_i)| < \epsilon$ or a certain number of iterations.

We remark some cases at this point. The first case is that there might be several local minima, that is, the algorithm does not guarantee getting the global minimum point in the given region. In this case, the initial point b_0 becomes important. Another case is that the global minimum might be on the boundary of $[\beta_0, \beta_1] \times [\alpha_0, \alpha_1]$ which might be a problem in implementing the gradient method. In our example we will take several initial values around the global minimum.

In the minimization of $F(b_i + \triangle b)$, we use the following estimate on $u(b_i + \triangle b)(t,0)$

$$u(b_i + \triangle b)(t,0) \simeq u(b_i)(t,0) + \nabla u(b_i)(t,0) \cdot \triangle b,$$

where ∇ denotes the gradient of $u(b)(t,0)$ with respect to b. $F(b_i + \triangle b)$ becomes

$$F(b_i + \triangle b) = \|\nabla u(b_i)(t,0) \cdot \triangle b + u(b_i)(t,0) - \varphi(t)\|_2^2 + \lambda \left\|\triangle b\right\|^2_E .$$

In numerical calculations, we note that $\|\cdot\|_2$ can be discretized by using a finite number of points in $[0,T]$, i.e., for $t_1 = 0 < t_2 < \cdots < t_q = T$, hence $F(b_i + \triangle b)$ has its new form as

$$E(\triangle b) \simeq \sum_{k=1}^{q} (u(b_i, t_k, 0) + \nabla u(b_i, t_k, 0) \cdot \triangle b - \varphi(t_k))^2 + \lambda \left\|\triangle b\right\|^2_E . \tag{4.2}$$

Now the minimization of this problem is a least squares problem whose solution leads to the following normal equation (see [7])

$$(\lambda I + A^T A)\triangle b = A^T K,$$

where

$$A = \left[\nabla u(b_i)(t_1, 0)^T \cdots \nabla u(b_i)(t_q, 0)^T\right],$$

and

$$K = \left[u(b_i)(t_1, 0) - \varphi(t_1) \cdots u(b_i)(t_q, 0) - \varphi(t_q)\right]^T.$$

Now the optimal direction is found by

$$\triangle b = (\lambda I + A^T A)^{-1} A^T K. \tag{4.3}$$

In forming A, the computation (or estimation) of s^{th} component of the vector $\nabla u(b_i)(t_k, 0)$ can be achieved by

$$\frac{u(b_i + he_s)(t_k, 0) - u(b_i + he_s)(t_k, 0)}{h}, \tag{4.4}$$

where e_s is the standard unit vector whose s^{th} component is 1 and h is the differential step. Now we give the algorithm.

Algorithm:

Step 1: Set b_0, λ and a stopping criterion k or ϵ (stop when the iteration number is equal to k or size of $\|\triangle b_i\|_E \le \epsilon$).
Step 2: Calculate $\triangle b_i$ using (4.3) and set $b_{i+1} = b_i + \triangle b_i$.
Step 3: Stop when the criterion is achieved.

5. Numerical examples with noise free and noisy data

In this section we examine the algorithm with two problems. Both problems are considered in the following form where (β, α) is to be found:

$$\begin{cases} \dfrac{\partial^\beta}{\partial t^\beta} u(t, x) = -(-\Delta)^{\alpha/2} u(t, x), \ -1 < x < 1, \ 0 < t < 1, \\ u(t, -1) = u(t, 1) = 0, \ 0 < t < 1, \\ u(0, x) = f(x), \ -1 < x < 1, \\ u(t, 0) = \varphi(t). \end{cases} \tag{5.1}$$

In both examples, the direct problem is solved for $a = (\beta, \alpha) = (0.5, 1)$ and the given $f(x)$ using (2.2) then $\varphi(t)$ is obtained numerically. Then the inverse problem with the obtained $\varphi(t)$ is solved via the algorithm to get $a = (\beta, \alpha) = (0.5, 1)$ for different (β, α) initial points.

The problems are solved first using noise-free data then noisy data and the results are established for both cases. We form the noisy data in the following way

$$\tilde{\varphi} = \varphi + \xi(t),$$

where $\xi(t) = \theta z(t)$ and $z(t)$ is a random number between [-1,1] and θ is noise level. For the noisy data, the optimal regularization parameters are sought and the results are established.

We recall that the solution of the direct problem is given

$$u(t,x) = \sum_{n=1}^{\infty} < f, \psi_n > E_\beta(-\lambda_n t^\beta)\psi_n(x),$$

where $\lambda_n = (\frac{n\pi}{2})^\alpha$ with $\psi_n(x) = \sin\left(\frac{n\pi x}{2}\right)$ when n is even and $\psi_n(x) = \cos\left(\frac{n\pi x}{2}\right)$ when n is odd, and

$$E_\beta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

Due to the fact that the solution of the direct problem involves infinite sums and requires the discretization of an integral functional $I(a)$, there is a large list of parameters used in the computations. These parameters and their values in our computations are listed below:

- The sensitivity of Mittag-Leffler function: 10^{-6} is taken.
- Summing index of $u(t,x)$: $r = 10$ is taken for the first example and $r = 20, 50, 100, 150$ are examined for the second example.
- The initial points for (β, α): are chosen to be the points of the form $(0.5, 1) + m(Re(e^{i\frac{k\pi}{8}}), Im(e^{i\frac{k\pi}{8}}))$ for $m = 0.2, 0.3$ and 0.4 and $k = 1, \cdots, 8$. Thus the algorithm is examined with 24 different points around $(0.5, 1)$.
- The number of t_i's in $[0,1]$ for in (4.2): $q = 10$ is taken.
- Stepsize h in (4.4): $h = 0.1$ is taken.
- Stop criterion: $\|\triangle b_i\| < 0.01$ in (4.3) or 100 iterations.
- Regularization parameters λ: are obtained for the perturbed data in both examples heuristically. In both examples, the best regularization parameter λ is chosen among multiples of 0.01 in [0,1] by comparing the relative error for each.
- Noise level θ: is taken as 0.1 for both examples.

The relative error for each result corresponding to an initial point is computed as the following

$$\frac{\|u(\beta,\alpha)(t,x) - u(1,0.5)(t,x)\|_\infty}{\|u(1,0.5)(t,x)\|_\infty},$$

where (β, α) is the corresponding result for the initial point and $\|\cdot\|_\infty$ is taken to be the maximum value of the absolute value of $u(t,x)$ at 20000 points formed by the Cartesian product of 200 uniformly distributed points on $[-1,1]$ and 100 uniformly distributed points on $[0,1]$ for x and t respectively.

Example 1. $f(x) = 0.5\cos(2.5\pi x) + \sin(\pi x)$. Note that this choice of $f(x)$ allows $< f, \psi_n >= 0$ except $n = 2$ and 5. For this reason, the summing index has been chosen to be $r = 10$ for this example. See Table 1 for the noise-free data and Table 2 for the noisy data. In Table 3, the initial point

TABLE 1. The results obtained from 24 different points around $\beta = 0.4, \alpha = 1$ using the noise-free data.

initial values of (α, β)	results with $r = 10$	initial values of (α, β)	results with $r = 10$
1.2000 0.4000	0.9995 0.4003	1.3000 0.4000	1.0000 0.4006
1.1732 0.5000	0.9994 0.3999	1.2598 0.5500	0.9992 0.3996
1.1000 0.5732	1.0001 0.4007	1.1500 0.6598	0.9996 0.4001
1.0000 0.6000	0.9996 0.4003	1.0000 0.7000	0.9996 0.4001
0.9000 0.5732	0.9996 0.4004	0.8500 0.6598	0.9996 0.4001
0.8268 0.5000	0.9995 0.4002	0.7402 0.5500	0.9996 0.4006
0.8000 0.4000	0.9995 0.4000	0.7000 0.4000	0.9994 0.3999
0.8268 0.3000	0.9996 0.4000	0.7402 0.2500	0.9994 0.3998
0.9000 0.2268	0.9997 0.4000	0.8500 0.1402	0.9996 0.4001
1.0000 0.2000	0.9997 0.4001	1.0000 0.1000	0.9998 0.4000
1.1000 0.2268	0.9996 0.4004	1.1500 0.1402	0.9996 0.4001
1.1732 0.3000	0.9996 0.4001	1.2598 0.2500	0.9996 0.4005

TABLE 2. The results with relative errors obtained from 24 points around $\beta = 0.4, \alpha = 1$ using the noisy data with error level $\theta = 0.1$

initial values of (α, β)	results with $r = 10$	relative error
1.2000 0.4000	0.8677 0.6834	0.2984
1.1732 0.5000	0.8675 0.6826	0.2979
1.1000 0.5732	0.8679 0.6843	0.2989
1.0000 0.6000	0.8675 0.6829	0.2981
0.9000 0.5732	0.8683 0.6826	0.2977
0.8268 0.5000	0.8668 0.6794	0.2960
0.8000 0.4000	0.8664 0.6801	0.2966
0.8268 0.3000	0.8678 0.6842	0.2989
0.9000 0.2268	0.8676 0.6831	0.2982
1.0000 0.2000	0.8677 0.6839	0.2987
1.1000 0.2268	0.8678 0.6846	0.2991
1.1732 0.3000	0.8677 0.6838	0.2987
1.3000 0.4000	0.8670 0.6795	0.2960
1.2598 0.5500	0.8674 0.6824	0.2978
1.1500 0.6598	0.8673 0.6791	0.2957
1.0000 0.7000	0.8674 0.6823	0.2977
0.8500 0.6598	0.8675 0.6818	0.2974
0.7402 0.5500	0.8676 0.6818	0.2974
0.7000 0.4000	0.8677 0.6817	0.2973
0.7402 0.2500	0.8678 0.6820	0.2975
0.8500 0.1402	0.8677 0.6840	0.2988
1.0000 0.1000	0.8677 0.6835	0.2984
1.1500 0.1402	0.8680 0.6837	0.2986
1.2598 0.2500	0.8676 0.6819	0.2974

with the highest relative error in Table 2 is used for finding the best regularization parameter. The best regularization is found to be $\lambda = 0.05$ and the associated relative error is given in Table 3. Table 3 shows how important the regularization parameter is in fixing the results for the noisy data.

Table 3. The initial point with the highest relative error (See Table 2) is retested with regularization parameter $\lambda = 1.5$

initial values of (α, β)	results with $\lambda = 0$	relative error	results with $\lambda = 1.5$	relative error
1.1000 0.2268	0.8678 0.6846	0.2991	0.9217 0.3642	0.0194

Table 4. The results for 24 points around $\beta = 0.4, \alpha = 1$ for the noise-free data for different choices of summing indices

initial values of (α, β)	$r = 20$	$r = 50$	$r = 100$	$r = 150$
1.2000 0.4000	0.9909 0.4016	0.9909 0.4016	0.9999 0.4000	1.0249 0.3958
1.1732 0.5000	0.9909 0.4016	0.9909 0.4016	1.0000 0.4000	1.0249 0.3958
1.1000 0.5732	0.9909 0.4016	0.9909 0.4016	1.0000 0.4000	1.0250 0.3958
1.0000 0.6000	0.9909 0.4016	0.9910 0.4016	1.0000 0.4001	1.0250 0.3958
0.9000 0.5732	0.9909 0.4016	0.9910 0.4016	1.0000 0.4001	1.0250 0.3958
0.8268 0.5000	0.9909 0.4016	0.9909 0.4016	1.0000 0.4000	1.0249 0.3958
0.8000 0.4000	0.9912 0.4015	0.9912 0.4015	0.9999 0.4000	1.0249 0.3958
0.8268 0.3000	0.9910 0.4013	0.9910 0.4013	1.0002 0.3997	1.0249 0.3957
0.9000 0.2268	0.9907 0.4015	0.9907 0.4015	0.9997 0.3999	1.0249 0.3955
1.0000 0.2000	0.9909 0.4019	0.9909 0.4019	0.9999 0.4003	1.0247 0.3959
1.1000 0.2268	0.9909 0.4016	0.9909 0.4016	1.0000 0.4000	1.0250 0.3958
1.1732 0.3000	0.9909 0.4016	0.9909 0.4016	1.0000 0.4000	1.0249 0.3958
1.3000 0.4000	0.9909 0.4015	0.9909 0.4015	0.9999 0.3999	1.0249 0.3958
1.2598 0.5500	0.9909 0.4015	0.9909 0.4015	0.9999 0.3999	1.0249 0.3958
1.1500 0.6598	0.9909 0.4016	0.9909 0.4016	1.0000 0.4001	1.0249 0.3958
1.0000 0.7000	0.9909 0.4017	0.9910 0.4017	1.0000 0.4001	1.0251 0.3959
0.8500 0.6598	0.9909 0.4017	0.9909 0.4017	1.0000 0.4001	1.0250 0.3959
0.7402 0.5500	0.9909 0.4016	0.9909 0.4016	1.0000 0.4001	1.0249 0.3959
0.7000 0.4000	0.9909 0.4015	0.9909 0.4015	0.9999 0.4000	1.0249 0.3958
0.7402 0.2500	0.9909 0.4015	0.9909 0.4015	0.9999 0.3999	1.0249 0.3957
0.8500 0.1402	0.9910 0.4016	0.9910 0.4016	1.0000 0.4000	1.0250 0.3956
1.0000 0.1000	0.9909 0.4016	0.9910 0.4016	1.0000 0.4000	1.0249 0.3959
1.1500 0.1402	0.9910 0.4017	0.9910 0.4017	1.0000 0.4001	1.0251 0.3959
1.2598 0.2500	0.9909 0.4016	0.9909 0.4016	0.9999 0.4000	1.0249 0.3958

Example 2. $f(x) = e^{-10x^2}$. Note that not all $< f, \varphi_n >$ are strictly positive, however $< f, \varphi_0 >$ is strictly positive and it dominates the other values of $< f, \varphi_n >$. The algorithm is examined for $r = 20, 50, 100$ and 150. See Table 4 for the noise-free data and Table 5 for the noisy data. In examining the noisy data, $r = 100$ is used because of its relatively high precision in Table 4. In Table 6 the initial point with the highest relative error in Table 5 is used for finding the best regularization parameter. The best regularization parameter is found to be $\lambda = 0.33$ and the associated relative error is given in Table 3.

TABLE 5. The results with relative errors obtained from 24 points around $\beta = 0.4, \alpha = 1$ using the noisy data with error level $\theta = 0.1$

initial values of (α, β)	results with $r = 100$	relative error
1.2000 0.4000	0.9448 0.5101	0.0970
1.1732 0.5000	0.9448 0.5100	0.0970
1.1000 0.5732	0.9448 0.5100	0.0970
1.0000 0.6000	0.9450 0.5102	0.0970
0.9000 0.5732	0.9448 0.5102	0.0971
0.8268 0.5000	0.9449 0.5100	0.0969
0.8000 0.4000	0.9449 0.5098	0.0968
0.8268 0.3000	0.9447 0.5100	0.0970
0.9000 0.2268	0.9449 0.5101	0.0970
1.0000 0.2000	0.9448 0.5101	0.0970
1.1000 0.2268	0.9448 0.5101	0.0971
1.1732 0.3000	0.9448 0.5101	0.0970
1.3000 0.4000	0.9448 0.5101	0.0970
1.2598 0.5500	0.9448 0.5100	0.0970
1.1500 0.6598	0.9448 0.5100	0.0970
1.0000 0.7000	0.9449 0.5101	0.0970
0.8500 0.6598	0.9448 0.5101	0.0970
0.7402 0.5500	0.9448 0.5100	0.0970
0.7000 0.4000	0.9448 0.5100	0.0970
0.7402 0.2500	0.9448 0.5100	0.0970
0.8500 0.1402	0.9449 0.5101	0.0970
1.0000 0.1000	0.9448 0.5101	0.0970
1.1500 0.1402	0.9449 0.5103	0.0972
1.2598 0.2500	0.9448 0.5102	0.0971

TABLE 6. The initial point with the highest relative error (See Table 5) is retested with the regularization parameter $\lambda = 4.7$

initial values of (α, β)	results with $\lambda = 0$	relative error	result with $\lambda = 4.7$	relative error
1.1500 0.1402	0.9449 0.5103	0.0972	0.9835 0.3951	0.0030

From the computations above, we observe two important facts. The first one is that in the different neighborhoods of the correct value, the algorithm works well and this suggests that the functional $I(a)$ given by (3.14) is differentiable in some neighborhood of the mimimum. That conclusion is proved by Theorem 4.1 in Section 4 . We note that the requirement imposed on $< f, \phi_n >$ by Theorem 4.1 is roughly equivalent to the differentiability of $f(x)$ to some extent because $< f, \phi_n >$ is roughly equivalent to $\hat{f}(n)$ where $\hat{f}$ is the Fourier transform of f.

The other fact is that when the algorithm is applied to solve the inverse problems of the given type in this article for any $f(x)$, one should be careful about the summing index r of the direct problem for the requirement $< f, \phi_n >$ is strictly positive for every n is not satisfied for all functions and this yields considerable differences in different choices of r as seen in Table 4.

6. Concluding Remarks

We have studied a nonlocal inverse source problem for the space-time fractional diffusion $\frac{\partial^\beta}{\partial t^\beta}u(t,x) = -(-\Delta)^{\alpha/2}u(t,x)$. After defining the input-output mapping for the inverse problem, we have proved that the mapping is continuous. By using continuity of the mapping and compactness of the interval $[\beta_0,\beta_1]\times[\alpha_0,\alpha_1]$, we have concluded that the minimization problem has a solution. The uniqueness of the solution has been proved for a specific class of the initial functions $f(x)$ using eigenfunction expansion of the solution of the direct problem. For the numerical solution of the inverse problem, a numerical method based on discretization of the minimization problem, steepest descent method and least squares approach is proposed. The numerical algorithm determines the unknowns β and α simultaneously. In the future, we plan to study an inverse source problem for the nonhomogeneous space-time fractional diffusion equation $\frac{\partial^\beta}{\partial t^\beta}u(t,x) = -(-\Delta)^{\alpha/2}u(t,x) + F(t,x)$. After establishing existence and uniqueness of the solution of the inverse source problem theoretically, we will propose a numerical algorithm to determine $F(t,x)$. We also plan to determine simultaneously $F(t,x)$ and one of the parameters β or α. These are subjects of the future studies by the authors of this paper.

Acknowledgments

The research has been supported by the Scientific and Technological Research Council of Turkey (TUBITAK) through the project Nr 113F373, also by the Zirve University Research Fund. The authors thank the referees for their very careful reading and for pointing out several mistakes as well as for their useful comments and suggestions.

References

[1] M. Caputo. Linear models of diffusion whose Q is almost frequency independent, part II. *Geophys. J. R. Astron. Soc.*, 13:529–539, 1967. 2

[2] Q. Z. Chen, M. M. Meerschaert and E. Nane. Space-time fractional diffusion on bounded domains. *Journal of Math. Analysis and its Appl.*, 393:479–488, 2012. 2, 2

[3] J. Cheng, J. Nakagawa, M. Yamamoto and T. Yamazaki. Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. *Inverse Problems*, 25:115–131, 2009. 1

[4] S. D. Eidelman, S. D. Ivasyshen, A. N. Kochubei. Analytic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type. *Birkhäuser, Basel*, 2004. 2

[5] A. Hasanov, P. Duchateau, B. Pektas. An adjoint problem approach and coarse-fine mesh method for identification of the diffusion coefficient in a linear parabolic equation. *J. Inverse Ill-Posed Probl.*, 14:435–463, 2006. 3

[6] B. Jin, and W. Rundell. An inverse problem for a one-dimensional time-fractional diffusion problem. *Inverse Problems*, 28:075010, 2012. 1

[7] A. Kirsch. An Introduction to the Mathematical Theory of Inverse Problems *Springer*, 1996. 4

[8] Y. Liu, S. Tatar, S. Ulusoy, Quasi-solution approach for a two dimensional nonlinear inverse diffusion problem. *Applied Mathematics and Computation*, 219:10956–10960, 2013. 3

[9] J.J Liu and M. Yamamato. A backward problem for the time-fractional diffusion equation. *Applicable Analysis*, 89:1769-1788, 2010. 1

[10] Y. Luchko, Initial-Boundary-Value problems for the one-dimensional time-fractional diffusion equation. *Fractional Calculus Appl. Anal*, 15:141–160, 2012. 2, 2, 2

[11] Y. Luchko, Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation. *Computers and Mathematics with Applications* 59:1766–1772, 2010. 2

[12] K. H. Karlsen and S. Ulusoy. Stability of entropy solutions for Lévy mixed hyperbolic-parabolic equations. *Electron. J. Differential Equations*, 2011(116):1–23, 2011. 2

[13] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo. Theory and Applications of Fractional Differential Equations. *Elsevier, Amsterdam*, 2006. 2

[14] V. N. Kolokoltsov and M. A. Veretennikova Well-posedness and regularity of the Cauchy problem for nonlinear fractional in time and space equations. *Fractional Differential Calculus*, to appear, 2014. 1

[15] Y. Luchko, Multi-dimensional fractional wave equation and some properties of its fundamental solution. *ArXiv*, 2013. 2

[16] F. Mainardi, Y. Luchko and G. Pagnini. The fundamental solution of the space-time fractional diffusion equation. *Fractional Calculus and Applied Analysis*, 4:153-192, 2001. 1

[17] M. M. Meerschaert, D. A. Benson, H.-P. Scheffler and B. Baeumer. Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E*, 65, 2002. 1

[18] E.D. Nezza, G. Palatucci, and E. Valdinoci Hitchhiker's Guide to the Fractional Sobolev Spaces. *Bull. Sci. Math*, 136:521-573, 2012. 2

[19] Y.H. Ou, A. Hasanov, and Z. Liu. Inverse coefficient problems for nonlinear parabolic differential equations. *Acta Math. Sin. Engl. Ser*, 24:1617–1624, 2008. 3

[20] I. Podlubny. Fractional Differential Equations. *Academic Press, San Diego*, 1999. 1, 2, 2

[21] H. Pollard. The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$. *Bull. Amer. Math. Soc.* , 54:1115-1116, 1948. 2

[22] K. Sakamoto and M. Yamamato Inverse source problem with a final overdetermination for a fractional diffusion equation. *Mathematical Controls and Related Fields*, 4:509-518, 2011. 1

[23] K. Sakamoto and M. Yamamato Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems *Journal of Mathematical Analysis and Applications*, 382:426-447, 2011. 1, 2, 2, 2, 2, 3

[24] S. Tatar. Monotonicity of input-output mapping related to inverse elastoplastic torsional problem. *Applied Mathematical Modeling*, 37:9552–9561, 2013. 3

[25] S. Tatar, S. Ulusoy. A uniqueness result for an inverse problem in a space-time fractional diffusion equation. *Electron. J. Differential Equations*, 258:1–9, 2013. 2, 2, 2, 3

[26] X Xu, J. Cheng and M. Yamamato Carleman esimate for a fractional diffusion equation with half order and application. *Applicable Analysis*, 90:1355-1371, 2011. 1

[27] M. Yamamato and Y. Zhang Conditional stability in determining a zeroth-order coefficient in a half-order fractional diffusion equation by a Carleman estimate. *Inverse Problems*, 28:105010, 2012. 1

[28] Y. Zhang, and X. Xu. Inverse source problem for a fractional diffusion equation. *Inverse Problems*, 27:035010, 2011. 1

[29] G. Li, D. Zhang, X. Jia and M. Yamamato. Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time fractional diffusion equation. *Inverse Problems*, 29:065014, 2013. 1, 2, 3, 3

(Salih Tatar)
Department of Mathematics
Faculty of Education
Zirve University
Sahinbey, Gaziantep,
27260, Turkey
E-mail address: salih.tatar@zirve.edu.tr
URL: http://person.zirve.edu.tr/statar/

(Ramazan Tınaztepe)
Department of Mathematics
Faculty of Education
Zirve University
Sahinbey, Gaziantep,
27260, Turkey
E-mail address: ramazan.tinaztepe@zirve.edu.tr
URL: http://person.zirve.edu.tr/tinaztepe/

(Süleyman Ulusoy)
Department of Mathematics
Faculty of Education
Zirve University
Sahinbey, Gaziantep,
27260, Turkey
E-mail address: suleyman.ulusoy@zirve.edu.tr
URL: http://person.zirve.edu.tr/ulusoy/