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Riemann Solver for Hyperbolic Equations with Discontinuous Coefficients: A Mathematical Proof of the Constant State Formula

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ABSTRACT

In *Godunov* numerical methods type used in many industrial and scientific numerical simulations including; fluid dynamics, electromagnetic, electro-hydrodynamic problems, a *Riemann* problem needs to be solved to estimate fluxes. The exact solution is generally not possible to obtain, but good approximations are available, Roe and *HLLC Riemann* solvers are among the most popular. However, all these solvers assume that the acoustic waves speeds are continuous by considering some averaging. In a previous work the effect of such averaging is demonstrated to be significant for some applications leading to a wrong solution. A *Riemann* solver is proposed taking into account the discontinuity of the acoustic wave's speeds. The case that shows discrepancy comparing to the averaged solvers is the one with an acoustic wave's speeds having a negative left value and a positive right value. In this case a constant state appears and a formula of the constant state is proposed. A numerical, and a particular exact solution based on a regularization technique are provided to demonstrate the validity of the formula. However, and due to the important impact of this case on *Godunov* type schemes, a mathematical proof is necessary. In this paper the formula of the constant state is proved, the proof is based on the generalized functions algebra theory.

Keyword: Hyperbolic equations, *Riemann* solver, waves speed, *Godunov* scheme, CFD, generalized functions algebra.

INTRODUCTION

Numerical methods to solve a large range of PDEs, such as finite volume, Discontinuous Galerkin (DG), Discontinuous finite volume, require the estimation of numerical fluxes at cell (sub-cell) faces. The accuracy of the method depends on the accuracy of the flux estimation. For the convective fluxes, generally a *Riemann* problem is considered and then an approximation *Riemann* solver is used. This leads to a stable upwind numerical scheme. This approach was first proposed by *Godunov* [1], consequently such methods are referred to by *Godunov* type methods. Depending of the problem to solve, many *Riemann* solvers were developed. In computational fluid dynamics (CFD), the most popular being, the *Roe* solver [2,3], the HLL solver [7], and the HLLC solver [6]. For the *Roe* solver, the Jacobian matrix is averaged in such a way that hyperbolicity, consistency with the exact Jacobian and conservation across discontinuities still fulfilled. This solver has been modified [4,5], to overcome the shortcoming for low-density flows. The HLL solver, solves the original nonlinear flux to take nonlinearity into account. It has a major drawback however because of space averaging process, the contact discontinuities, shear waves and material interfaces are not captured. To remedy this problem, the HLLC solver was proposed by adding the missing wave to the structure. However, all these methods assume that the waves speed are continuous across the left and right states of the *Riemann* problem (through the cell interfaces of the mesh) by applying diverse averaging process. This is not true in general; typical situations are recirculation for turbulent flows and transitions from subsonic to supersonic for transonic regimes. The impact of this averaging on the obtained numerical methods has been demonstrated in [8,11,15]. A *Riemann* solver of scalar hyperbolic linear equation with discontinuous coefficient is developed, taking account the wave discontinuities. It is shown [15] that the case with a negative left value and positive right value of the wave speed, a constant state appears in the proposed *Riemann* solver. A wave propagation test case, shows a sensitive discrepancy comparing to the averaging-based scheme. This can be explained by a product of distributions that occurs, which is not defined by the classical theory of distributions. Note that for the other cases, no differer was observed between the *Godunov* scheme based on the proposed *Riemann* solver and the averaging solvers.

A formula of the constant state in the proposed *Riemann* solver was provided. Its validity is demonstrated through numerical tests and a particular exact solution based on regularization techniques. A mathematical proof is however necessary, which is the objective of this paper. Indeed, a proof based on the generalized functions algebra [9,13,14], is provided. In section 2, the proposed *Riemann* solver in [15] is described with the associated *Godunov* scheme for the linear case with the results showing the discrepancy. In section 3, an overview of the generalized functions algebra is provided. In section 4, a proof the the constant state is developed. Conclusions are drawn in section 5.

RIEMANN SOLVER FOR HYPERBOLIC EQUATION WITH DISCONTINUOUS COEFFICIENTS

Consider the *Cauchy* problem of a scalar linear hyperbolic equation with discontinuous coefficient,

$$\begin{aligned} \frac{\partial}{\partial t}\varphi + a(x)\frac{\partial}{\partial x}\varphi = 0, \quad \text{on } [o,T]\times\Omega \\ \varphi(0,x) = \varphi_0 \in L^\infty(\Omega) \\ a(.) \in L^\infty(\Omega) \end{aligned} \tag{1}$$

The initial condition $\varphi_0(x)$ and the coefficient $a(x)$ are bounded functions, and can be discontinuous. From the theoretical point of view, the problem is well-posed see in [9]. It is shown that the more critical case is when the solution φ and the coefficient $a(.)$ are discontinuous at the same location which leads to a product of distributions (for instance if $a(.)$ is some *Heaviside* function and a Dirac function resulting from the derivative of φ). This product is not defined in the classical space of distributions which is not an algebra. The well-posdeness of the problem is then studied in a more appropriate space of generalized functions introduced by *J.F Colombeau*, known as well as the *Colombeau's* algebra. For more details we refer to [9,13,14]. Now, let's define the *Riemann* problem associated with problem (1)

$$\frac{\partial}{\partial t}\varphi + a(x)\frac{\partial}{\partial x}\varphi = 0, \quad \text{on } [o,T]\times\Omega$$
$$\varphi(0,x) = \varphi_0 = \begin{cases}\varphi_L & \text{if } x<0\\ \varphi_R & \text{if } x>0\end{cases} \tag{2}$$
$$a(x) = \begin{cases}a_L & \text{if } x<0\\ a_R & \text{if } x>0.\end{cases}$$

In this equation, the acoustic wave speed $a(.)$ is discontinuous, which again, is not taken into account in the existing *Riemann* solvers where acoustic waves speed is averaged. In [8,15], a *Riemann* solver is proposed based on the following observations of different possible situations:

Case 1
$a_L > 0$ and $a_R > 0$ we have propagation of the discontinuity (of initial condition) to the right and we do not need to consider what happening within the fan defined by the two acoustic waves, because they will catch up if $a_L > \beta_R$ and if $a_L < \beta_R$ an expansion will appear.

Case 2
$a_L < 0$ and $a_R < 0$ similar to the previous case with a propagation of the discontinuity to the left.

Case 3
$a_L < 0$ and $a_R > 0$ we have propagation of the discontinuity to the left and the right simultaneously, and we need to determine what happened within the fan defined by the two acoustics waves. We assume that a constant state appears, and its expression will be given below.

Case 4:
$a_L > 0$ and $a_R < 0$ in this case we have opposite acoustic waves speed and then the discontinuity will remain blocked, which means there is no propagation.

Based on the above observations, the *Riemann* solution of problem (2) is given by

$$\varphi(x,t) = \begin{cases}\varphi_L & \text{if } a_L > 0 \ \textit{and} \ a_R > 0\\ \lambda & \text{if } a_L < 0 \ \textit{and} \ a_R > 0\\ \varphi_R & \text{if } a_L < 0 \ \textit{and} \ a_R < 0\\ \varphi^0 & \text{if } a_L > 0 \ \textit{and} \ a_R < 0\end{cases} \tag{3}$$

Where the expression of the constant λ is given by

$$\lambda = \frac{\frac{1}{|a_L|}\varphi_L + \frac{1}{|a_R|}\varphi_R}{\frac{1}{|a_L|} + \frac{1}{|a_R|}} \tag{4}$$

***Godunov* Scheme**

The *Godunov* numerical schemes for the linear scalar hyperbolic equations with discontinuous coefficients using the *Riemann* solver (3)-(4) is given by. Let h and Δt be the space and time steps, then set $x_{i-1/2} = (i-1/2)h$ and $t_n = n\Delta t$, and

$$a_i^n = a(t_n, x_i) \qquad \varphi_i^n = \varphi(t_n, x_i) \tag{12}$$

The *Godunov* scheme approximation is then given by

$$\varphi_i^{n+1} = \left(\varphi_{i-1/2}^{n+1,R} + \varphi_{i+1/2}^{n+1,L}\right)/2 \tag{13}$$

where $\varphi_{i-1/2}^{n+1,L}$ and $\varphi_{i-1/2}^{n+1,R}$ are obtained by the classical projection process and given by.

Case $a_{i-1}^n < 0$ and $a_i^n > 0$

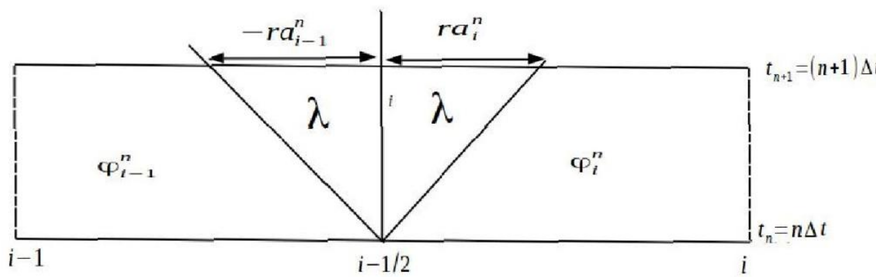


Figure 6: Riemann Solution: Case of intermediate constant

$$F_1 = \begin{cases} \varphi_{i-1/2}^{n+1,R} = \varphi_i^n - 2ra_i^n\left(-\lambda_{i-1/2}^n + \varphi_i^n\right) \\ \varphi_{i-1/2}^{n+1,L} = \varphi_{i-1}^n - 2ra_{i-1}^n\left(\lambda_{i-1/2}^n + \varphi_{i-1}^n\right) \end{cases} \tag{14}$$

with

$$\lambda_{i-1/2}^n = \frac{\dfrac{1}{|a_{i-1}^n|}\varphi_{i-1}^n + \dfrac{1}{|a_i^n|}\varphi_i^n}{\dfrac{1}{|a_{i-1}^n|} + \dfrac{1}{|a_i^n|}}$$

Case $a_{i-1}^n < 0$ and $a_i^n < 0$

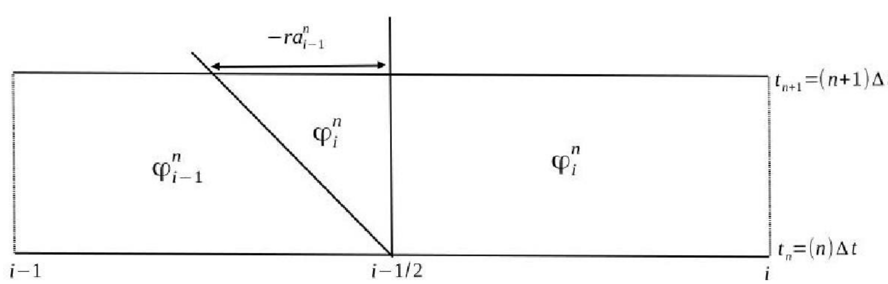


Figure 7: Riemann Solution: Case of negative speed

$$F_2 = \begin{cases} \varphi_{i-1/2}^{n+1,R} = \varphi_i^n \\ \varphi_{i-1/2}^{n+1,L} = \varphi_{i-1}^n - 2ra_{i-1}^n(\varphi_i^n - \varphi_{i-1}^n) \end{cases} \tag{15}$$

Case $a_{i-1}^n > 0$ and $a_i^n < 0$

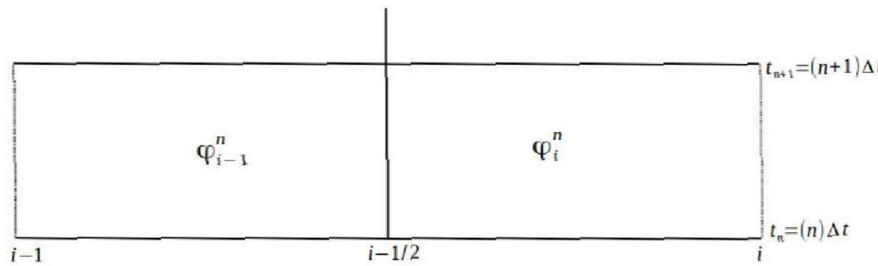


Figure 8: Riemann Solution: Case of blocked wave

$$F_3 = \begin{cases} \varphi_{i-1/2}^{n+1,R} = \varphi_i^n \\ \varphi_{i-1/2}^{n+1,L} = \varphi_{i-1}^n \end{cases} \tag{16}$$

Case $a_{i-1}^n > 0$ and $a_i^n > 0$

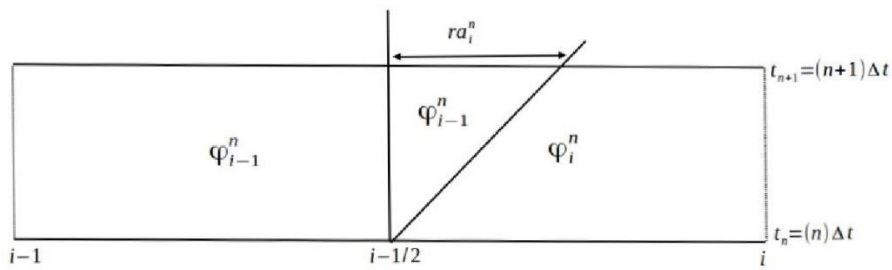


Figure 9: Riemann Solution: Case of positive speed

$$F_4 = \begin{cases} \varphi_{i-1/2}^{n+1,R} = \varphi_i^n - 2ra_i^n(\varphi_i^n - \varphi_{i-1}^n) \\ \varphi_{i-1/2}^{n+1,L} = \varphi_{i-1}^n \end{cases} \tag{17}$$

For the case where the waves speed is averaged, the *Godunov* scheme is obtained in the same way with formulas reduced to only two cases as a result of the averaging process, namely we have;

$$Case\ \ of\ \ \alpha_{i-\frac{1}{2}}^n = 0.5(a_i^n + a_{i-1}^n) > 0$$

$$F_1 = \begin{cases} \varphi_{i-1/2}^{n+1,R} = \varphi_i^n - 2r\alpha_{i-\frac{1}{2}}^n(\varphi_i^n - \varphi_{i-1}^n) \\ \varphi_{i-1/2}^{n+1,L} = \varphi_{i-1}^n \end{cases} \tag{18}$$

$$Case\ \ of\ \ \alpha_{i-\frac{1}{2}}^{n} = 0.5(a_i^n + a_{i-1}^n) < 0$$

$$F_2 = \begin{cases} \varphi_{i-1/2}^{n+1,R} = \varphi_i^n \\ \varphi_{i-1/2}^{n+1,L} = \varphi_{i-1}^n - 2r\alpha_{i-\frac{1}{2}}^{n}(\varphi_i^n - \varphi_{i-1}^n) \end{cases}$$

Numerical Tests

In [15] the following test case is performed to assess the wave averaging process impact. consider the coefficient $a(.)$ to be a two states function

$$a(x) = \begin{cases} -2 & if x < 0 \\ 3 & if x > 0. \end{cases} \tag{20}$$

Here the case with the intermediate constant state λ in the proposed *Riemann* solver is considered. The other cases show no difference comparing the waves averaged numerical scheme. The initial condition is first taken to be a sinusoidal function (smooth function), to avoid a product of distribution of type *Heaviside* times a Dirac function (not defined in the classical space of distributions) which was the main motivation to the use of the *Colombeau* algebra for the mathematical analyse of the problem. Results are depicted in Fig.1 and Fig.2. As we can see both schemes provide the same results, this means that in this case averaging waves speed allows too well capture the intermediate state with a correct value.

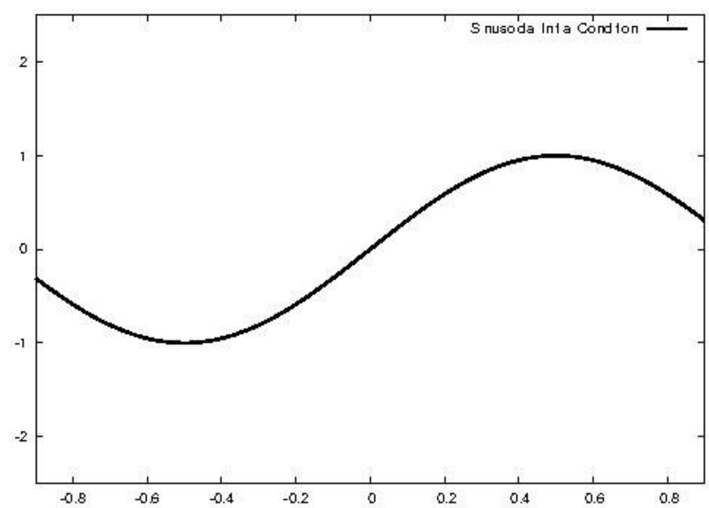
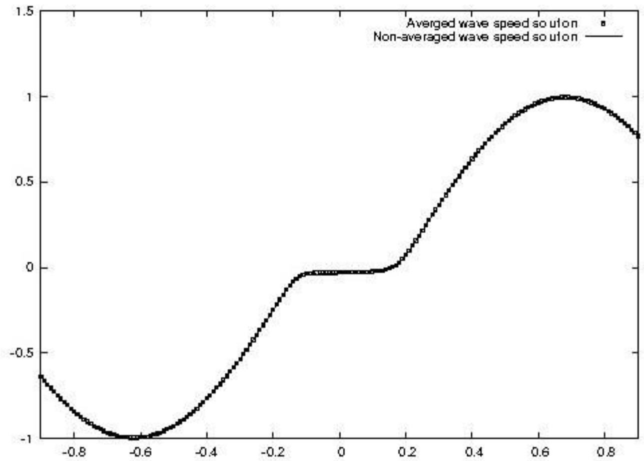
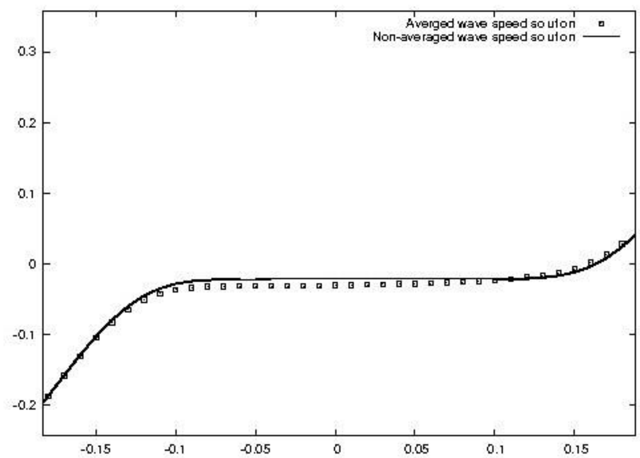


Figure 1: Sinusoidal initial condition



(a)



(b)

Figure 2: (a) Propagation of the initial sinusoidal condition using the proposed Riemann solver and the averaged on, (b) Zoom on the plateau (the intermediate constant)

In figures Fg.3 to Fig.8, a sinusoidal and polynomial initial condition with a discontinuity at the origin were considered in [15]. The results show a sensitive discrepancy between the two solutions. The scheme obtained by waves speed averaging is capable to predict the plateau (the intermediate state) but with a completely wrong value. Even if a numerical and a particular exact solution is used to justify the formula (4) of the intermediate constant appearing in the proposed scheme suggesting that results obtained with the later are the correct ones, a mathematical proof still necessary for a confirmation. After a short overview of the generalized functions algebra a proof of the formula is provided in the following sections.

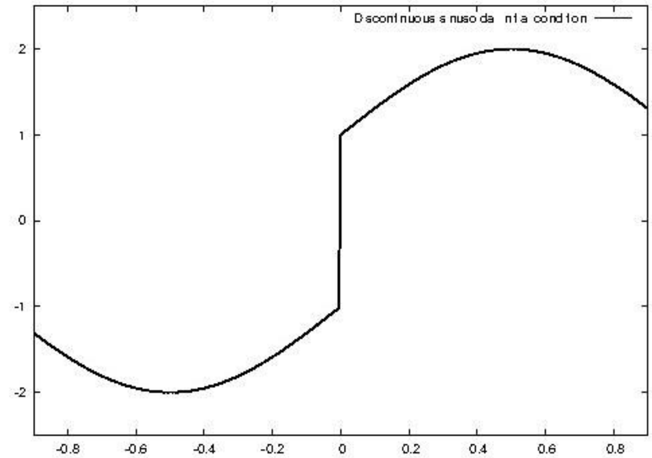
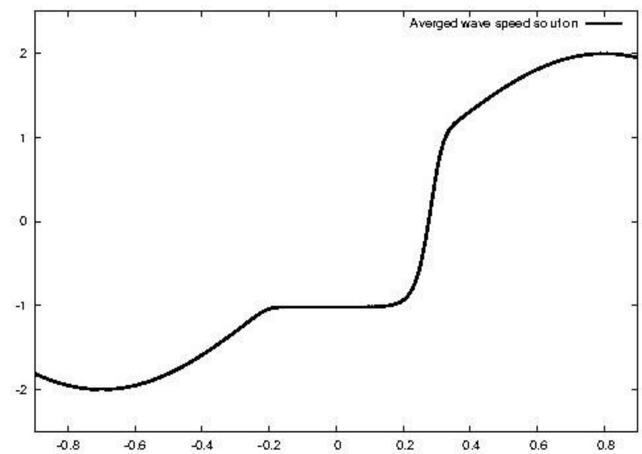
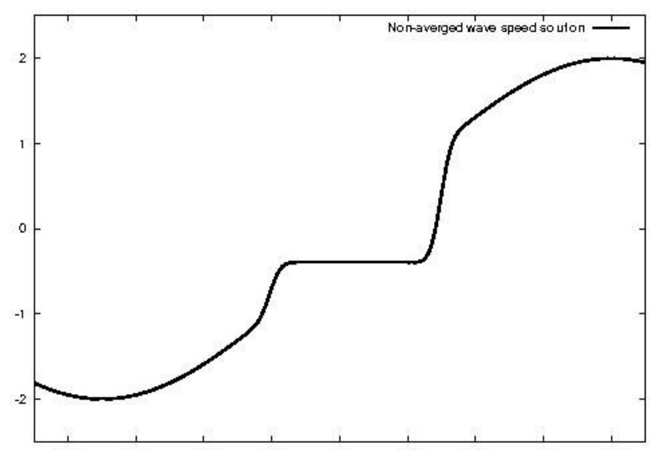


Figure 3: Discontinuous sinusoidal initial condition



(a)



(b)

Figure 4: (a) Propagation of the initial sinusoidal condition using the averaged Riemann solver, (b) Propagation of the initial sinusoidal condition using the proposed Riemann solver

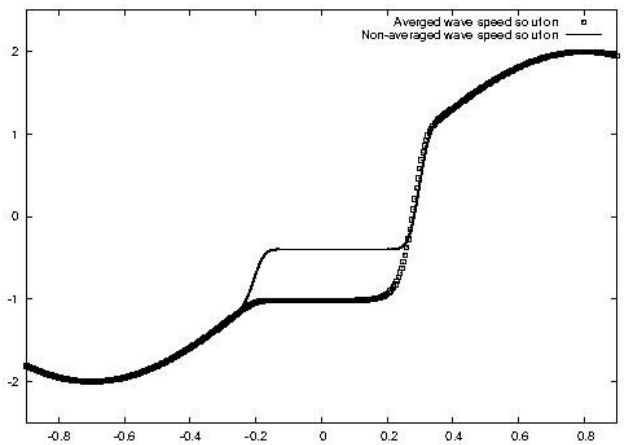


Figure 5: Curves of Figure (4) superimposed.

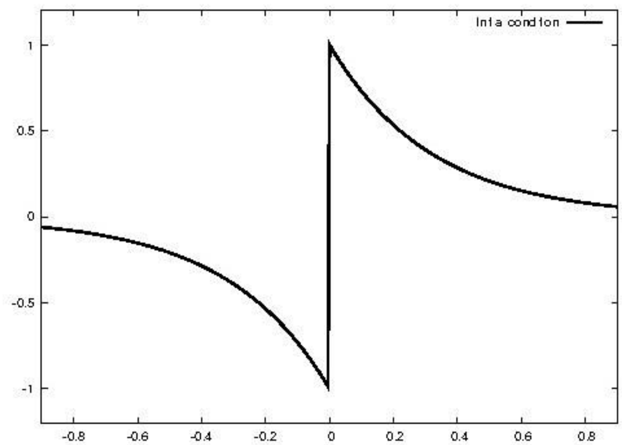
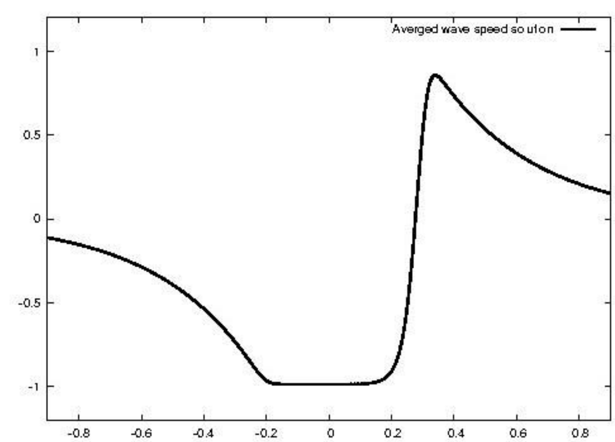
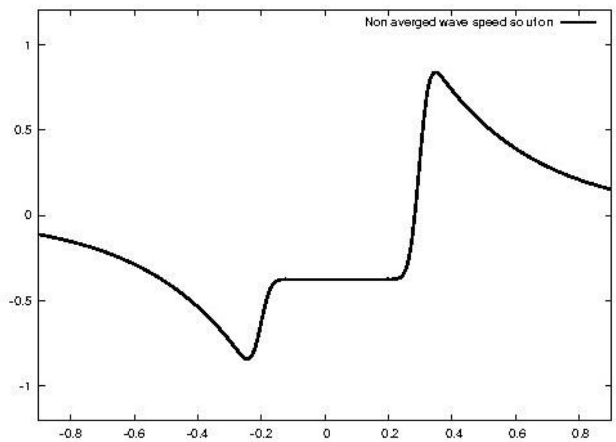


Figure 6: Discontinuous polynomial initial condition



(a)



(b)

Figure 7: (a) Propagation of the initial polynomial condition using the averaged Riemann solver, (b) Propagation of the initial polynomial condition using the proposed Riemann solver

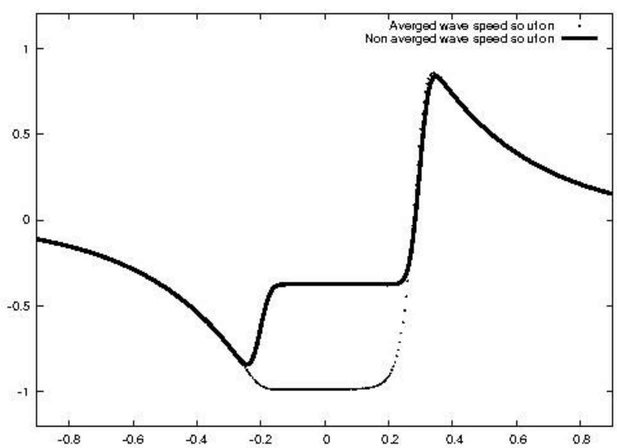


Figure 8: Curves of Figure (7) superimposed

OVERVIEW OF THE GENERALIZED FUNCTION SPACE

The following is an overview of the generalized function algebra, also known as the *Colombeau* algebra. For details about this theory, see [13].

Definition 1

For $q=0,1,2,\ldots$ set $A_q=\{\varphi\in D(\Re^n)\ \ such\ \ that\ \int\varphi dx=1\, and \int x^j\varphi dx=0\ \ if\ \ 1\le|j|\le q\}$.Define by $E(\Omega)$ the set of functions R such that:

$$R: A_1\times\Omega\rightarrow\Re$$
$$(\Phi,x)\rightarrow R(\Phi,x)$$

where R is a C^∞ function on x for each fixed Φ

Proposition 1

The set $E(\Omega)$ is an algebra and $C^\infty(\Omega)$ is a sub-algebra of $E(\Omega)$.

Definition 2

We define by $E_M(\Omega)$ the sub-algebra of $E(\Omega)$ functions that have a moderate increase. This means that $E_M(\Omega)=$

$$R: \textit{such that for each compact set } K, \textit{ for each derivative of order } p, \textit{ there exists an integer } N$$

$$\left\{ \textit{such that if } \Phi \in A_N, \textit{ then there exists } \eta > 0, C > 0 \textit{ such that}: \sup_{x\in K} \left| R_\varepsilon^{(p)}(\Phi, x)\right| \le C(\frac{1}{\varepsilon})^N \quad \textit{with } 0<\varepsilon<\eta<1 \right\},$$

Where,

$$R_\varepsilon(\Phi, x) = R(\Phi_\varepsilon, x) \text{ and } \Phi_\varepsilon(\lambda) = \frac{1}{\varepsilon^n}\Phi(\frac{\lambda}{\varepsilon}).$$

Definition 3

Let $\mathrm{I}(\Omega)=$

$$\left\{ \begin{array}{l} R \in E_M(\Omega) \textit{ such that for any compact set } K \textit{ of } \Omega, \textit{ for any derivative of order } p, \textit{ there exists} \\ \textit{an integer } N \textit{ such that for each function } \Phi \in A_N, \textit{ there exists a constant}, C>0 \textit{ and } \eta>0, \textit{ such that} \\ \textit{for each sufficiently large } q \textit{ we have}: \sup_{x\in K}\left|R_\varepsilon^p(\Phi, x)\right| \le C(\frac{1}{\varepsilon})^{q-N} \textit{ with } 0<\varepsilon<0. \end{array} \right\}$$

Proposition 2

The set $I(\Omega)$ is a vectorial subspace and an ideal of $E_M(\Omega)$.

The generalized functions algebra is defined by,

Definition 4

The C^∞ generalized functions algebra $G(\Omega)$ is defined as the quotient:

$$G(\Omega) = \frac{E_M(\Omega)}{I(\Omega)}$$

The above definition of the generalized functions algebra is rather abstract and, practically-speaking, how to handle the elements of $G(\Omega)$ is not clear. Thus, a simplified space of the generalized functions algebra (G_s) is defined, without a canonical inclusion of the distributions. Refer to Chapter 8 of [3] regarding this immediate simplification.

If Ω is any open set in R^n, a space of "simplified global generalized functions" is defined as follows.

The reservoir of representatives is:

$$E_s(\Omega) = \{all\, maps\, \Re \in C^{\infty}(]0,1] \times \Omega, R),\ such\ that, \forall D\,(partial\ x\, derivative, including\, the\, identity)$$
$$\exists N \in IN, c > 0,\ such\ that: \forall x \in \Omega, \left|(DR)(\varepsilon, x)\right| \le \frac{c}{\varepsilon^N}\},$$

and the ideal $N_s(\Omega)$ is given by:

$$N_s(\Omega) = \{R \in C^{\infty}(\Omega),\ such\ that: \forall D, \forall q \in IN\ \exists c_q > 0,\ such\ that: \forall x \in \Omega, \left|(DR)(\varepsilon, x)\right| \le c_q \varepsilon^q\}.$$

Then the space $G_{s,g}(\Omega)$ of the simplified global generalized functions on Ω is the quotient algebra $G_{s,g}(\Omega) = \dfrac{E_s(\Omega)}{N_s(\Omega)}$. The term "global" is employed in this instance because the above bounds hold globally on Ω, and not only on compact subsets of Ω, as in [13].

The Association Concept:
G_1, $G_2 \in G_s(\Omega)$ are said to be associated (we write $G_1 \approx G_2$), if and only if, for any Ψ in $D(\Omega)$, we have:

$$\int_{\Omega} \left[R_1(\varepsilon, x) - R_2(\varepsilon, x)\Psi(x)\right] dx \to 0\ when\ \varepsilon \to 0$$

where $R_1 \in E_s$ is a representative of G_1 and $R_2 \in E_s$ is a representative of G_2.

The association could be viewed as a weak generalization of the equality concept. Consequently, when dealing with PDEs, the equality is replaced by the association unless the equality is justified.

The Product of Generalized Functions:
The product of two generalized functions is defined naturally by the class of the product of its representatives (by replacing the equality with the association concept) i.e., if G_1, $G_2 \in G_s(\Omega)$ and R_1, R_2 are their respective representatives, then $G_1 . G_2 \approx$ Class { $R_1 . R_2$ }. A non-linear regular function of generalized functions is defined in more general terms in [13,14].

Now we will examine which classical functions or distributions can be represented by generalized functions.

Inclusions:

Let f be a function in the space $D_{L^\infty}(\Omega), i.e., f$ is a C^∞ function on Ω, globally bounded on Ω as well as its derivatives; then with f, associate $R(\varepsilon, x) = f(x)$. This gives the inclusion $D_{L\infty}(\Omega) \subset G_s(\Omega)$. Let f be a function in the space $L^\infty(IR^n)$; then with f, associate $R(\varepsilon, x) = f * \rho_\varepsilon(x)$ with a chosen $\rho \in D(IR^n)$, $\int \rho(\lambda)d\lambda = 1$ and $\rho_\varepsilon(\lambda) = \frac{1}{\varepsilon^n}\rho\left(\frac{\lambda}{\varepsilon}\right)$. For any given mollifier ρ, this gives an inclusion $L^\infty(IR^n) \subset G_s(IR^n)$. In more general terms, let T be a distribution in $D'_{L\infty}(IR^n)$, i.e., T is a finite sum of the derivatives of functions in $L^\infty(IR^n)$. With T, associate $R(\varepsilon,x) = (T * \rho_\varepsilon)(x)$ as above. Therefore, for a given ρ, this provides an inclusion of $D'_{L\infty}(IR^n)$ in $G_s(IR^n)$. Similarly, there is an inclusion of $E'(\Omega)$ — the space of all distributions with compact support — in $G_s(\Omega)$. All these inclusions become canonical, i.e. the arbitrariness in the choice of a mollifier ρ disappears if we work in the space $G(\Omega)$ of "non-simplified" generalized functions, exactly as in Chapter 8 of [6], whose definition is slightly more complex.

Proposition 3

Let $T \in D'(\Omega)$. Then there exists $g \in G_{s,g}(\Omega)$ associated with T. We write $g \approx T$ and say that T is the "macroscopic aspect" of g. (The proof is given in [13, 14].)

Derivatives of Generalized Functions:

A derivative of a generalized function (∂g) is naturally defined by the class of equivalence of a given representative. That is, if $R(\varepsilon, x)$ is a representative of g, ∂g = Class $\partial R(\varepsilon, x)$.

Regularized Derivatives:

The concept of regularized derivatives ($\bar{\partial}$) constitutes the basic ingredient ensuring existence and uniqueness results in many situations. The properties of these derivatives are discussed in [16, 17]; we recall, for instance, that if g is a generalized function then $\bar{\partial} g$ and ∂g are associated in $G_{s,g}$. We shall now give the definition of $\bar{\partial}$: If $R(\varepsilon, x)$ is a representative of a generalized function g, if ∂ is a partial x-derivative, if $\rho : \Re^n \to \Re$ with $\int \rho(\mu)d\mu = 1$ ($\rho \in D(\Re^n), or\ \rho\ step\ function$) is a "mollifier" and if $h : \varepsilon \to h(\varepsilon)$ is a scaling function ($h : [0,1] \to [0,1]\ and\ h(\varepsilon) \to 0\ when\ \varepsilon \to 0$), then the mollified derivative $\tilde{\partial}^h g$ is the class of $\left(\partial R(\varepsilon,.) * \rho_{h(\varepsilon)}\right)(x)$.

PROOF OF THE λ FORMULA

Now consider the *Riemann* problem within the framework of generalized functions algebra. This wrote

$$\begin{aligned} \frac{\partial}{\partial t}\varphi + a(x)\frac{\partial}{\partial x}\varphi = 0, \quad \text{on } [o,T] \times \Omega \\ , \qquad \varphi(0,x) = \varphi_0 = \varphi_L + \Delta\varphi Y(x) \\ a(x) = a_L + \Delta a L(x) \end{aligned}$$

With

$$\Delta\varphi = \varphi_R - \varphi_L$$
$$\Delta a = a_R - a_L$$

Where Y and L are two representatives of the *Heaviside* function. A solution in the case of the intermediate constant λ can be set on the form:

$$\varphi(t,x) = \varphi_L + (\lambda - \varphi_L)H_\varepsilon(x - a_L t) + (\varphi_R - \lambda)K_\varepsilon(x - a_R t)$$

We can check that $\varphi(t,x)$ is indeed a solution.

Then, on any time interval included in $[\frac{1}{a_L}, \frac{1}{a_R}]$ the function φ is constant equal to λ. Then $\frac{\partial}{\partial t}\varphi(t,x) = 0$ on this interval

$$-a_L(\lambda - \varphi_L)H_\varepsilon{}'(x - a_L t) - a_R(\varphi_R - \lambda)K_\varepsilon{}'(x - a_R t) = 0$$

Taking the limit $t \to 0$, we get

$$-a_L(\lambda - \varphi_L)H_\varepsilon{}'(x) - a_R(\varphi_R - \lambda)K_\varepsilon{}'(x) = 0$$

Multiply both sides by $\Psi \in D(\Omega)$ the integrate with respect of x

$$-a_L(\lambda - \varphi_L)\int H_\varepsilon(x)\Psi'(\mathrm{x})\mathrm{dx} - a_R(\varphi_R - \lambda)\int K_\varepsilon(x)\Psi'(\mathrm{x})\mathrm{dx} = 0 \quad (12)$$

Since H_ε and $K_\varepsilon$are representatives of Heaviside function

$$\lim_{\varepsilon\to 0}\int H_\varepsilon(x)\Psi'(\mathrm{x})\mathrm{dx} = \lim_{\varepsilon\to 0}\int K_\varepsilon(x)\Psi'(\mathrm{x})\mathrm{dx}$$

Taking the limit on ε in (12)

$$-a_L(\lambda - \varphi_L)\lim_{\varepsilon\to 0}\int H_\varepsilon(x)\Psi'(\mathrm{x})\mathrm{dx} - a_R(\varphi_R - \lambda)\lim_{\varepsilon\to 0}\int H_\varepsilon(x)\Psi'(\mathrm{x})\mathrm{dx} =$$
$$[-a_L(\lambda - \varphi_L) - a_R(\varphi_R - \lambda)]\lim_{\varepsilon\to 0}\int H_\varepsilon(x)\Psi'(\mathrm{x})\mathrm{dx} = 0$$

We can always choose Ψ such that $\lim_{\varepsilon\to 0}\int H_\varepsilon(x)\Psi'(\mathrm{x})\mathrm{dx} \neq 0$ which means that

$$-a_L(\lambda - \varphi_L) - a_R(\varphi_R - \lambda) = 0$$

This lead to

$$(-a_L + a_R)\lambda = -a_L\varphi_L + a_R\varphi_R$$

And then

$$\lambda = \frac{-a_L\varphi_L + a_R\varphi_R}{-a_L + a_R}$$

Since we are in the case $a_L < 0$ and $a_R > 0$ we can write

$-a_L = |a_L|$ and $a_R = |a_R|$ then

$$\lambda = \frac{|a_L|\varphi_L + |a_R|\varphi_R}{|a_L| + |a_R|}$$

By dividing both numerator and denominator by $|a_L||a_R|$ we obtain the formula (4)

$$\lambda = \frac{\frac{1}{|a_R|}\varphi_L + \frac{1}{|a_L|}\varphi_R}{\frac{1}{|a_R|} + \frac{1}{|a_L|}}$$

Which ends the proof.

CONCLUSIONS

The paper dealt with a mathematical proof within the framework of the generalized functions algebra of a constant state formula proposed in a previous work to build a *Riemann* solver for hyperbolic linear equations with discontinuous coefficients. It is demonstrated in the previous work that *Godunov* type numerical schemes built on waves speed averaging, like Roe and HLLC schemes, are capable to capture the constant state but with a wrong value, which can have a serious impact on numerical methods used in many applications like CFD (Navier-Stokes equations), Electromagnetism (Maxwell's Equations), etc. Only numerical and particular solution are provided as a justification of the validity of the proposed formula in the previous work. Due to the great importance of such discrepancy, a mathematical proof was necessary, which is provided in this paper, to encourage and convince researchers in all fields, where *Riemann* solvers are used, to consider constant state case. This can explain some discrepancies, observed in many numerical simulations, with experimental results.

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