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Review

Short-term spot rate models with nonparametric deterministic drift

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ABSTRACT

Most studies assume stationarity when testing continuous-time interest-rate models. However, consistent with Bierens [Bierens, H. (1997). Testing the unit root with drift hypothesis against nonlinear trend stationary, with an application to the US price level and interest rate. *Journal of Econometrics*, *81*, 29–64; Bierens, H. (2000). Nonparametric nonlinear co-trending analysis, with an application to interest and inflation in the United States. *Journal of Business and Economics Statistics*, *18*, 323–337], our nonparametric test results support nonlinear trend stationarity. To accommodate nonstationarity, we detrend the interest-rate series and re-examine a variety of continuous-time models. The goodness-of-fit improves significantly for those models with drift-induced mean reversion and worsens for those with high volatility elasticity. The inclusion of a nonparametric trend component in the drift significantly reduces the level effect on the interest-rate volatility. These results suggest that the misspecification of the constant elasticity model should be attributed to the nonlinear trend component of the short-term interest-rate process.

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1. Introduction

In continuous-time finance, the drift and diffusion functions of a Markov process fully characterize the dynamics of the short-term interest rate. As noted in Chan, Karolyi, Longstaff, and Sanders (1992), hereafter CKLS, the most important feature differentiating interest-rate models is the volatility elasticity specified in the diffusion functions. The interest-rate volatility and its dependence on the level of interest rates are critical elements in the valuation of interest-rate contingent claims and the design of optimal strategies to hedge against interest-rate risk. Yet, the appropriate specification of this volatility function remains, for the most part, an unanswered question.

The recent evidence on the alternative parametric specifications of the diffusion functions has provided mixed results. CKLS and Conley, Hansen, Luttmer, and Scheinkman (1997) find that the volatility of interest-rate changes is highly sensitive to the level of the interest rate. Both studies find evidence supporting volatility elasticity greater than one. On the other hand, Koedijk, Nissen, Schotman, and Wolff (1997), Andersen and Lund (1997), Hong, Li, and Zhao (2004), and Cristiansen (2005) document a substantially lower 'level effect'. They allow for conditional heteroscedasticity in the diffusion function of the interest rate and find that the volatility elasticity is not significantly different from 0.5, which is in accordance with the Cox, Ingersoll, and Ross (1985), hereafter CIR SR model. Their findings indicate that the inclusion of a 'volatility effect' considerably reduces the level effect.

Another stream of related studies identify nonlinearities in the interest rate. Using semi-nonparametric approach, Aït-Sahalia (1996) finds strong nonlinearity in the drift function of the interest rate. The same conclusion is also documented by applying a fully nonparametric approach (e.g. Conley et al., 1997; Jiang, 1998; Stanton, 1997). A growing amount of research is directed at the issue. For example, Pritsker (1998) adjust the finite sample properties of Aït-Sahalia's test by taking into account the high persistence of the interest rate. He finds that the evidence supports nonlinearity of the interest rate disappears. Using simulation analysis, Chapman, Long, and Pearson (2000) provide evidence that even when the true data generating process has a linear drift, the nonparametric tests used in the above studies may still generate nonlinear estimates for the drift function. Using the data transformation method to correct for the boundary bias of the kernel estimators, Hong and Li (2005) strongly reject a variety of one-factor diffusion models and conclude that the inclusion of a nonlinear drift does not improve the models' performance. In contrast, Arapis and Gao (2006) provide an evidence that the specification of the drift has a considerable impact on the pricing of derivatives through its effect on the diffusion function. In addition, they propose a specification test of linearity in the drift. Their results reject the null hypothesis of linearity in the drift for the short-term interest rate.

The estimation methods used in most of these studies assume that interest rates are stationary. However, several studies provide evidence of nonstationarity in the short-term interest-rate data. Aït-Sahalia (1996) and Bandi (2002) finds that the short-term interest rate is a unit-root process and Bierens

(1997, 2000) documents nonlinear trend stationarity. We note therefore that the a priori assumption of stationary interest rates generally used in continuous-time empirical finance can lead to inaccurate inference and incorrect conclusions.

To address this issue, this paper re-examines a wide variety of well-known models for the short-term interest rates without assuming stationarity up front. We first conduct extensive parametric and nonparametric diagnostic tests to determine whether the short-term interest rate is stationary, linear trend-stationary, nonlinear trend-stationary, or a unit-root process. In the case of nonlinear trend stationarity, we use the Hamming (1973) and Bierens (1997), henceforth HB, filtering approach, to locate the stationary component in the interest-rate process. We employ the generalized method of moments to test the each model's ability to capture this stationary component. In addition, we incorporate the trend component in the drift function and examine its impact on the volatility elasticity and the mean-reverting behavior of interest rates. To account for structural breaks, we perform tests of unknown change points by using the techniques developed by Andrews (1993) and Andrews and Ploberger (1994) and re-estimate the models for the several sub-periods.

Our main findings can be summarized as follows. First, our diagnostic tests indicate that the short-term interest rate is nonlinear trend stationary, which confirms findings in Bierens (1997, 2000). Second, for the stationary component of the interest rates, the goodness-of-fit tests show a substantial improvement in those interest-rate models that allow for drift-induced mean reversion and a dramatic worsening in the performance of those that allow for high volatility elasticity. Third, the introduction of a nonlinear trend-stationary component in the drift function significantly reduces the level effect in the diffusion function. These results suggest that the high-level effect in the previous empirical studies can be attributed mainly to the nonlinear trend component of the short-term interest-rate process. In addition, our sub-sample analysis shows that all mean-reverting models outperform the non-reverting models and that the level effect and its impact on each model's performance differ considerably across sub-periods.

The remainder of the paper is constructed as follows. Section 2 reviews the models to be considered. Section 3 outlines the data used. In Section 4, we describe the diagnostic tests, the estimation method, the HB decomposition procedure, and the structural-break tests. In Section 5, we discuss the results. Section 6 concludes.

2. The interest-rate models

As in CKLS, we consider a stochastic differential equation that encompasses a broad class of interest-rate processes,

$$dr(t) = [\alpha + \beta r(t)]\,dt + \sigma r^{\gamma} dZ(t), \tag{1}$$

where $r(t)$ is the spot interest rate and $Z(t)$ is a standard Brownian motion. We denote the stochastic differential Eq. (1) as the unrestricted interest-rate model. We consider nine parametric special cases of this general constant elasticity volatility (CEV) model. The specification of each nested model can easily be obtained by setting restrictions on the parameters α, β, γ and σ. The first special case under consideration is the Merton (1973) model, which imposes the parameter restrictions $\beta = \gamma = 0$. Under the Merton model, we note that the interest-rate process is a Brownian motion with drift.

The second case, the Vasicek (1977) interest-rate model, requires the parameter restriction $\gamma = 0$. Under the Vasicek model, the interest rate is an Ornstein–Uhlenbeck process with a restoring drift that pushes it downwards when the process is above α/β and upwards when it is below. The third model is the CIR SR interest-rate process, where the interest-rate volatility depends on the square root of the interest-rate level ($\gamma = 0.5$). This process ensures positive interest rates. The fourth model appears in Dothan (1978) and Brennan and Schwartz (1977). This model is nested in the general model (1) with parameter restrictions $\alpha = \beta = 0$ and $\gamma = 1$. Thus, the Brennan–Schwartz model is a driftless interest-rate process that allows the conditional volatility to be proportional to the interest-rate level.

The fifth model, used in Marsh and Rosenfeld (1983), imposes the parameter restrictions $\alpha = 0$ and $\gamma = 1$. This model for the interest rate is simply the geometric Brownian motion (GBM) introduced by Black and Scholes (1973). The sixth model, considered in Brennan and Schwartz (1980), is mean-reverting and has a volatility elasticity, γ, equal to one. The seventh model is used in Cox, Ingersoll,

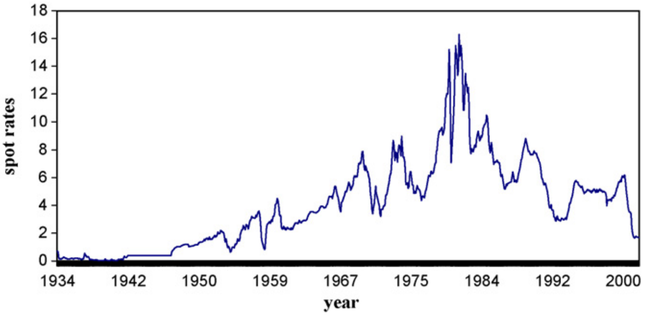

Fig. 1. The short-term interest-rate series in levels. The data is the three-month Treasury bill rate from the Federal Reserve Economic Database (FRED) over the sample period from January 1934 to July 2002 (826 annualized monthly observations).

and Ross (1980), henceforth CIR VR, and can be nested in the unrestricted model (1) by letting $\alpha = \beta = 0$ and $\gamma = 1.5$. Under this model, the interest-rate process is driftless and displays high level effects in its conditional volatility. The eighth model, henceforth the CEV model, is the restricted constant elasticity of variance process introduced by Cox (1975) and Cox and Ross (1976). This process only constrains the long-term mean parameter, α, to be equal to zero.

3. The data

We use three-month Treasury bill yield data that cover the period from January 1934 to July 2002 and are collected from the Federal Reserve Economic Database (FRED) of the Federal Reserve Bank of St. Louis. The data is monthly and the total number of observations is 826. We use the three-month Treasury bill rate as a proxy for the short-term interest rate to avoid microstructure problems or institutional features related to other money market instruments. As shown in Chapman, Long, and Pearson (1999), the proxy problem when using the three-month Treasury bill rate is trivial in the case of single-factor interest-rate models. In addition, the three-month Treasury bill rate has been used recently in the term structure literature [e.g., Andersen and Lund (1997) and Stanton (1997)]. The full series of this three-month interest rate is displayed in Fig. 1.

Table 1 shows the means, standard deviations, and first five autocorrelations of the three-month yield and the monthly changes in the three-month yield. The unconditional average of the three-month yield is 4.01% with a standard deviation of 3.21%. We see that the autocorrelations in both the level and monthly changes decay significantly and are not consistently positive or negative.

4. The methodology

This section briefly describes the diagnostic tests, the generalized method of moments (GMM) of Hansen (1982), the HB decomposition procedure, and the structural break tests of Andrews (1993) and Andrews and Ploberger (1994).

Table 1
Descriptive statistics for the three-month Treasury bill rate.

Variables	N	Mean	Standard deviation	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
r_t	826	0.004	0.032	0.992	−0.633	0.249	−0.038	0.013
$r_{t+1} - r_t$	825	0.000	0.003	0.407	−0.227	0.032	−0.029	0.036

This table reports the means, standard deviations, and autocorrelations of monthly yields and yield changes for the three-month Treasury bill. The sample period is from January 1934 to July 2002. N is the total number of observations, r_t denotes the monthly yields, $r_{t+1} - r_t$ represents the monthly changes in the yields, and ρ_j is the autocorrelation of order j.

4.1. Diagnostic tests

In order to determine whether the interest rate is stationary, trend-stationary, nonlinear trend-stationary, or a unit-root process, we employ several unit-root tests. First, we test the null hypothesis of unit root against the alternative of stationary and linear trend stationary using the traditional Augmented Dickey–Fuller (ADF) test, the Phillips and Perron (1988) (PP) test, and the Augmented Weighted Symmetric (WS) test. Pantula, Gonzalez-Farias, and Fuller (1994) argue that the WS test is a weighted, double-length regression that dominates the ADF and PP tests in terms of power. The Akaike Information criterion (AIC) is used to determine the optimal number of lags for all the tests. We use MacKinnon (1994) approximation, which is robust to size distortion, to compute the ADF, PP, and WS asymptotic p-values.

Second, we employ the non-parametric method of Breitung (2002) to test for unit root against the alternative hypotheses of stationarity and linear trend stationarity. Monte Carlo simulations show that the Breitung test is robust to structural breaks. Finally, we employ Bierens (1997) tests where the unit root with drift hypothesis is tested against the alternative of nonlinear trend stationarity. The $t(m)$, $A(m)$, and $T(m)$ Bierens tests are described in Appendix A. As shown in Appendix A, the Bierens tests are based on an Augmented Dickey–Fuller auxiliary regression with linear and nonlinear deterministic trends. For the nonlinear trend, Bierens uses transformed Chebishev polynomials that are orthogonal to time. The null hypothesis of interest for these tests is a unit root with drift. However, a rejection of the null does not necessarily imply that the process is nonlinear trend stationary. For example, for the $t(m)$ test statistic, a left-sided rejection of the null does not provide information on the alternative. In contrast, the model-free $T(m)$ has the power to distinguish between three hypotheses; a left-sided rejection of the null suggests linear trend stationarity; and a right-sided rejection points in the direction of nonlinear trend stationarity.

4.2. Goodness-of-fit tests

Following Brennan and Schwartz (1982), Dietrich-Campbell and Schwartz (1986), CKLS, and others, we use the following discrete-time econometric specification to estimate the parameters of the continuous-time interest-rate models:

$$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{t+1} \tag{2}$$

$$E[\varepsilon_{t+1}] = 0, \qquad E[\varepsilon_{t+1}^2] = \sigma^2 r_t^{2\gamma}. \tag{3}$$

We employ the generalized method of moments (GMM) of Hansen (1982) to estimate the parameters in (2) and (3) simultaneously. The GMM procedure is widely used to estimate and test interest-rate models (see for example, CKLS, Harvey, 1988, and Longstaff, 1989). This method has relevant characteristics for the case at hand. First, the GMM procedure does not rely on any distribution assumption. This is an attractive feature for our analysis since not all the models under consideration have closed-form density functions. Second, the variance–covariance matrix in the GMM procedure is consistent with heteroscedastic and serially correlated residuals.

The estimation procedure involves minimizing the GMM criterion function, which is the quadratic form given by

$$g_T'(\theta) W_T(\theta) g_T(\theta), \tag{4}$$

where $g_T(\theta)$ is the vector of the moment conditions, $W_T(\theta)$ is the positive-definite symmetric weighting matrix suggested by Hansen (1982), and $\theta = (\alpha, \beta, \sigma^2, \gamma)'$ is the vector of parameters. For comparison purposes, we choose the same moment restrictions as in CKLS,

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} f_t(\theta), \tag{5}$$

where

$$f_t(\theta) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1} r_t \\ \varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma} \\ (\varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma}) r_t \end{bmatrix}. \tag{6}$$

We use the minimized value of the quadratic form in (4) to perform goodness-of-fit tests for each model. This test statistic is χ^2 distributed with degrees of freedom equal to the number of orthogonality conditions minus the number of parameters to be estimated.

4.3. The HB decomposition procedure

Using the method in Hamming (1973) and Bierens (1997), we decompose the short-term interest rate,

$$r_t = c_t + m_t \tag{7}$$

where c_t is the stationary component and m_t is the nonlinear trend function. The trend component m_t can be written as

$$m_t = \sum_{j}^{n-1} \gamma_{j,n} P_{j,n}(t), \quad t = 1, 2, \ldots, n, \quad \text{where } \gamma_{j,n} = (1/n) \sum_{t=1}^{n} m_t P_{j,n}(t), \tag{8}$$

where $P_{j,n}(t)$ is Chebishev time polynomials takes the form

$$P_{0,n}(t) = 1, \qquad P_{j,n}(t) = \sqrt{2} \cos\left[\frac{j\pi(t-0.5)}{n}\right], \quad j = 1, \ldots, n-1. \tag{9}$$

which are orthogonal to time.

After filtering the interest-rate data and rescaling, we use the GMM estimation and specification tests described above to examine each model's ability to capture the stationary component of the interest rates. This treatment also allows us to assess the impact of nonstationarity on the performance of the competing models. For robustness, we also use an alternative detrending approach. We modify the unrestricted model and the well-known CIR SR by incorporating the nonlinear trend component, m_t, in the drift function. This modification takes most of the time trend out of the innovation term and, consequently, provides a better understanding of the relation between volatility elasticity and the performance of short-term interest-rate models.

4.4. Structural break tests

The structural break tests introduced by Andrews (1993) and Andrews and Ploberger (1994) allow for robust tests of parameter instability without assuming exogenous break points. These tests are designed to test the hypothesis of no structural change against the alternative of one structural break. Consequently, to examine multiple breaks, it will be more powerful to do the test several times by excluding the periods after the breaks.

In the case of tests of one-time structural change, the null and alternative hypothesis of interest are respectively H_0: $\delta = 0$ and H_a: $\delta \neq 0$, where λ is the pre-break parameter vector and $\lambda + \delta$ the post-break parameter vector. Based on the Wald statistics, $W_T(\pi)$, Andrews (1993) construct the following 'Supremum' (*Sup-W*$_T$) test statistic:

$$\sup_{\pi \in \Pi} W_T(\pi), \tag{9}$$

where $\Pi = [\pi_1, \pi_2]$ is the time interval under consideration and π is the change point in this set. For optimality, Andrews and Ploberger (1994) design the more general 'Exponential' Wald test statistics

($Exp\text{-}W_T$) for structural breaks,

$$Exp\text{-}W_T = \frac{1}{\pi_2 - \pi_1 + 1} \sum_{\pi=\pi_1}^{\pi_2} \exp[0.5 W_T(\pi)]. \tag{10}$$

The $Exp\text{-}W_T$ in (10) is an exponentially weighted test statistic that gives more power to alternatives for which δ is large. For robustness, we also consider the 'Average' (*Ave*) Wald test statistics,

$$Ave\text{-}W_T = \frac{1}{\pi_2 - \pi_1 + 1} \sum_{\pi=\pi_1}^{\pi_2} W_T(\pi), \tag{11}$$

which is an equally weighted test statistic that is designed for alternatives that are very close to the null hypothesis. Following Andrews (1993) and Andrews and Ploberger (1994), we allow the trimming parameter to be between 15% and 30% of the effective sample. More recently, Hansen (1997) develops approximation methods to calculate p-values for the Andrews (1993) and Andrews and Ploberger (1994) tests. The hypothesis of no structural breaks in the entire parameter vector is rejected if the p-values based on Hansen's approximation method are below 5%.

5. The results

For comparison purposes, this section first estimates the nine models by a priori assuming stationarity and absence of structural breaks in the interest-rate process. Next, extensive diagnostic tests determine whether the interest rate is stationary, linear trend-stationary, nonlinear trend-stationary, or a unit-root process. Based on these test results, we examine how the models fit the stationary component of the interest-rate series and compare these results with those obtained for the raw, aggregate interest-rate series. To examine the robustness and implications of our results, we estimate modified versions of the unrestricted CEV and CIR SR models that displays mean-reversion toward a nonlinear dynamic trend. We also test for structural breaks in the aggregate interest-rate process and re-estimate the nine models for each sub-period.

5.1. Full-sample estimation and test results

5.1.1. The case of raw, aggregate interest-rate series

Table 2 reports the parameter estimates and goodness-of-fit tests for the models when applied to the full sample of aggregate interest-rate series. As in CKLS, we see that the ranking of the interest-rate models can be primarily classified by the volatility elasticity parameter, γ. At a conventional 5 percent significance level, the GMM criterion tests reject all the short-term interest-rate models with $\gamma < 1$, but cannot reject most models with $\gamma \geq 1$. The two models with the highest volatility elasticity, that is, the

Table 2
Full-sample GMM estimators and goodness-of-fit using unfiltered interest-rate series.

Model	α	β	σ^2	γ	χ^2
Unrestricted	0.0331 (0.267)	−0.0079 (0.440)	0.0004 (0.185)	1.649 (1.68e−16)	
Merton	0.0081 (0.301)	0.0	0.0493 (2.65e−18)	0.0	10.748 (0.0046)
Vasicek	0.0030 (0.912)	−0.0021 (0.824)	0.0489 (6.14e−18)	0.0	11.031 (0.0009)
CIR SR	0.0039 (0.988)	−0.0033 (0.723)	0.0174 (2.89e−21)	0.5	8.407 (0.0037)
Dothan	0.0	0.0	0.0036 (2.23e−23)	1.0	6.194 (0.103)
GBM	0.0	0.0032 (0.236)	0.0036 (9.58e−23)	1.0	4.967 (0.0835)
Brennan–Schwartz	0.0105 (0.701)	−0.0003 (0.976)	0.0036 (5.04e−23)	1.0	4.920 (0.0265)
CIR VR	0.0	0.0	0.0006 (3.23e−23)	1.5	2.663 (0.447)
CEV	0.0	0.0038 (0.255)	0.0005 (0.213)	1.592 (1.12e−13)	1.221 (0.269)

χ^2 is the test-statistic for the GMM overidentifying restrictions of each interest-rate model. The degree of freedom is the number of parameter restrictions that each nested model imposes on the unrestricted model. The numbers in parentheses are the p-values. The full-sample period is from 1934 to 2002.

CEV (γ = 1.592) and the CIR VR (γ = 1.5), outperform all the other models, whereas the Vasicek model (γ = 0) with a *p*-value of 0.0009 ranks last.

The estimated value for γ is about 1.6 for the unrestricted and CEV models, and is highly significant for both models. This result for γ is consistent with the findings in CKLS and Bandi (2002). We note, however, that there is a trade-off between γ and σ^2. The higher the value for γ, the lower the estimated value for the instantaneous volatility parameter, σ^2. In this respect, we see that the value of σ^2 is the lowest for the unrestricted model and the highest for the Merton model (γ = 0). The trade-off is not only with respect to the value of these parameters, but also with respect to their statistical significance. For all the models with γ as a fixed parameter, we observe that the parameter σ^2 is estimated very precisely and has a *p*-value almost equal to zero, whereas it is statistically insignificant in the unrestricted and CEV models.

As to the drift function, we note that the parameter estimates do not provide a clear-cut insight into the behavior of the drift. Consistent with CKLS, we find that all the estimates for the drift parameters α and β are statistically insignificant, and economically negligible. For the Vasicek, CIR SR, Brennan–Schwartz, and unrestricted models, this result is often interpreted as evidence against mean-reversion in the short-term interest rate. However, since the drift parameters are also insignificant across the models that do not imply mean-reverting interest rates, we suggest caution when drawing such a conclusion.

A comparison of the goodness-of-fit of models with (approximately) the same volatility elasticity, γ, shed more light on the role of the drift specification. Within each group, we notice that *p*-values rank the driftless models first, the non-reverting models (either α or β equal to zero) second, and the mean-reversion models last. This result suggests that the short-term interest rate behaves as a martingale over most of the full-sample period, which is broadly consistent with the findings in Aït-Sahalia (1996), Bandi (2002) and Stanton (1997). For example, Aït-Sahalia and Bandi find that the drift is virtually zero over most of the range of the short-term interest rate (3–15%) and mean-reverts only when it approaches the upper bound of its range.

Overall, we see that the driftless CIR VR model with its high level effect (γ = 1.5) outperforms all the competing models and provide a reasonable fit for the interest-rate dynamics over the full-sample period. Bandi (2002) find similar results for the CIR VR model using daily seven-day Eurodollar deposit rates over the period 1973–1995. Given that this model implies nonstationary behavior for the short-term interest rate, its superior performance raises concerns about the stationarity of the interest-rate series.

5.1.2. Diagnostic tests

Table 3 reports the results of the stationarity tests for the short-term interest rate. In Panel A, the conventional ADF and PP tests systematically fail to reject the null hypothesis of a unit root with only a constant term. Different choices of the lag length do not affect these results. Since these conventional tests lack power against the alternative hypothesis of stationarity, we use the more powerful WS test and the nonparametric Breitung test to verify these results. Similar to the conventional tests, the WS and Breitung tests fail to reject the unit root with drift hypothesis. The failure to reject unit root with drift in the interest-rate process corroborates with the findings in Aït-Sahalia (1996) and Bandi (2002).

In Panel B, we present the test results for a unit root with drift and a linear trend. The ADF, PP, WS, and Breitung tests indicate that the null hypothesis of a unit-root process for the short-term interest rate cannot be rejected in favor of a stationary or linear trend-stationary process. Although these results suggest modeling interest rates as unit-root processes over their full range, we should be cautious for the following reasons. First, if the nominal interest rate follows a random walk with a positive drift it would converge to infinity, which is not realistic. Second, if the nominal interest rate is a driftless random walk then it allows for negative values, which is not plausible from an economic point of view. Third, as noted in Bierens (1997), the unit-root hypothesis may prevail because some trend-stationary and unit-root processes show a remarkable similarity. The tests considered above only take one special case of trend stationarity into account, and ignore trend break and nonlinear trend stationarity. For a closer examination of the nonstationarity of the short-term interest rate, we consider the more general trend stationarity alternative hypothesis designed by Bierens (1997).

Table 3
Nonstationarity test results for the short-term interest rates.

Test	Test statistic	p-value
Panel A: Unit-root tests with a constant		
ADF	−2.242	0.466
PP	−12.398	0.295
WS	−2.445	0.327
Breitung	0.0104	0.421[a]
Panel B: Unit-root tests with a constant and a linear trend		
ADF	−2.045	0.170
PP	8.389	0.199
WS	−2.141	0.228
Breitung	0.0587	0.386[a]

Panel C: Bierens tests of unit-root with drift against nonlinear trend stationarity

Test	Test statistic	5% critical value	Simulated p-value
$t(m)$	−3.985	−3.971	0.421
$A(m)$	−40.22	−27.20	0.049
$T(m)$	326.78	280.57	0.040

ADF: Augmented Dickey–Fuller test; PP: Phillips–Perron test; and WS: Weighted Symmetric test. The lag length is equal to 19. We implement the nonstationarity tests for the level of the short-term interest rate covering the period from 1934 to 2002. The null hypothesis of a unit root is rejected if the p-value is smaller than 0.05. The ADF, PP, and WS asymptotic p-values are computed using MacKinnon (1994) approximation, which is robust to size distortion.
The Beirens $t(m)$, $A(m)$ and $T(m)$ tests are described in Section 4.1. $m = 10$ is the order of the Chebishev polynomial. For the Bierens $T(m)$ test, a right-sided rejection of the null hypothesis of unit root is an indication of nonlinear trend stationarity. We use simulated p-value based on 1000 replications drown from the normal distribution with zero mean and OLS squared residuals variances (the Wild bootstrap).

[a] For the Breitung test, we use simulated p-value based on 1000 replications drown from the normal distribution with zero mean and OLS squared residuals variances (the Wild bootstrap).

Panel C reports the $t(m)$, $A(m)$, and $T(m)$ test statistics for the unit-root null hypothesis against a nonlinear trend stationarity. We see that all these tests reject the unit-root null hypothesis at the 5% level on the basis of the asymptotic critical values. The value of −3.985 for the t-test statistic implies a left-sided rejection of the null hypothesis. As pointed out by Bierens (1997), for the t-test statistic, a left-sided rejection of the null does not provide information about the nature of the alternative. In this respect, we note that the $T(m)$ test statistic is more informative. In particular, we find that the $T(m)$ test statistic rejects the null on the right-side, which is an indication that nonlinear trend stationarity is the alternative. Consistent with Bierens (1997, 2000), we conclude therefore that the short-term interest rate is nonlinear trend stationary.

5.2. The case of detrended interest-rate series

Given the nonlinear trend stationarity of the interest-rate series, we use the HB-filter to detrend the series and examine each model's ability to capture the stationary component of the interest rate. Table 4 reports the parameter estimates and goodness-of-fit tests for the models applied to the full sample of filtered interest-rate series. We notice that the ranking of the model performances is no longer based on the high volatility elasticity, but rather on the drift specification of each model. The goodness-of-fit of all the models with mean-reverting drift improve considerably. The overidentifying restrictions of the mean-reverting Brennan–Schwartz and CIR SR models cannot be rejected at the conventional 5% level. In contrast, there is considerable worsening in the performance of the driftless models. The p-value of the goodness-of-fit statistic for the Dothan, CEV and CIR VR model decreases dramatically from 0.103, 0.269 and 0.447 to 0.505, 0.0011 and 0.0078, respectively, implying strong rejection of the last two models. This result suggests that for the stationary series a mean-reverting drift is the most important feature of the interest-rate model.

Table 4
Full-sample GMM estimators and goodness-of-fit using detrended interest-rate series.

Model	α	β	σ^2	γ	χ^2
Unrestricted	0.0007 (0.02111)	−0.0175 (0.09179)	0.05380 (0.5432)	1.3869 (0.0000)	
Merton	0.0002 (0.0383)	0.0	0.0000 (0.0000)	0.0	7.7277 (0.0209)
Vasicek	0.0002 (0.3680)	−0.0016 (0.8544)	0.0000 (0.0000)	0.0	7.7793 (0.0052)
CIR SR	0.0007 (0.1792)	−0.0057 (0.5247)	0.0002 (0.0000)	0.5	4.6130 (0.0317)
Dothan	0.0	0.0	0.0052 (0.0000)	1.0	7.7921 (0.0505)
GBM	0.0	0.0053 (0.0952)	0.0051 (0.0000)	1.0	4.8200 (0.0898)
Brennan–Schwartz	0.0005 (0.0476)	−0.0120 (0.1978)	0.0053 (0.0000)	1.0	1.3239 (0.2498)
CIR VR	0.0	0.0	0.0953 (0.0000)	1.5	9.1009 (0.0279)
CEV	0.0	0.0053 (0.0960)	0.0049 (0.6395)	0.9933 (0.0046)	4.8077 (0.0283)
Modified unrestricted	0.0002 (0.6470)	−0.0002 (0.9803)	0.0057 (0.42956)	1.05024 (0.00002)	
Modified CIR SR	0.0003 (0.5636)	0.0011 (0.9139)	0.0002 (0.0000)	0.5	11.2354 (0.18871)

χ^2 is the test-statistic for the GMM overidentifying restrictions of each interest-rate model. The degree of freedom is the number of parameter restrictions that each nested model imposes on the unrestricted model. The interest-rate data is detrended using 10 Chebishev time polynomials filtering technique. The numbers in parentheses are the *p*-values. The full-sample period is 1934–2002.

When comparing the performance within the mean-reverting class of interest-rate models, we observe that the level effect still plays an important role in the performance of the models. The *p*-value ranks the Brennan–Schwartz ($\gamma = 1$) first, CIR SR ($\gamma = 0.5$) second, and Vasicek ($\gamma = 0$) last in the mean-reverting group. Within the non-reverting models, we find that a higher value for γ does not necessarily improve the performance of the model. For example, the CEV model with an estimate of 0.993 for γ has a higher *p*-value than the CIR VR ($\gamma = 1.5$). In addition, we see that for the unrestricted and CEV models the volatility elasticity decreases from 1.649 and 1.592 to 1.386 and 0.993, respectively, when we detrend the interest rate. This result suggests that for the stationary series a mean-reverting drift with unitary elasticity is the most important feature of the interest-rate model.

Our findings suggest that the evidence on high volatility elasticity and weak drift-induced mean reversion documented in the literature is primarily due to the nonlinear trend component of the short-term interest rate. The a priori assumption of stationarity when specifying and testing interest-rate models implicitly shifts all the nonstationarity in the innovations and makes the drift specification less relevant for the overall performance of the models. To verify our results, we introduce the nonlinear trend component, m_t, in the drift function of the CIR SR and the unrestricted model.[1] This alternative detrending procedure allows us to take (most of) the nonstationarity out of the diffusion term.

The last two rows of Table 4 report the results for the nonlinear trend-stationary models of short-term interest rates. The goodness-of-fit test shows substantial improvement for the nonlinear trend-stationary CIR SR model relative to the standard counterpart. Specifically, the *p*-value of the goodness-of-fit statistic sharply increases from 0.003 to 0.188 for the CIR SR model. In addition, we see that for the unrestricted model the volatility elasticity decreases from 1.649 to 1.050 when we accommodate for trend stationarity.

5.3. Sub-sample analysis

5.3.1. Structural break tests

Table 5 presents the results of the structural break tests for the unrestricted model of interest-rate dynamics. The *p*-values of the *Sup*-W_T statistics in Fig. 2 suggest three potential structural changes. The first eventual change point is at the beginning of 1970, the second is mid 1974, and the third occurs between October 1979 and October 1981, which is commonly known as the 'Monetary Experiment' period of the Federal Reserve System. Following Andrews (1993), we only consider that point where

[1] We choose only two models to keep the presentation of the results manageable. We select the CIR SR model because it is the most popular mean-reverting model and ensures positive nominal interest rate. The CEV model allows us gauge the impact of the nonlinear trend component on the volatility elasticity.

Table 5
Test results for parameter stability and structural breaks with unknown change points.

Period	Test statistics			Max date
	Sup-W_T	*Ave*-W_T	*Exp*-W_T	
January 1934–July 2002	28.369 (0.000)	5.747 (0.0215)	8.052 (0.000)	April 1981
January 1934–April 1981	12.204 (0.0262)	5.567 (0.0332)	3.940 (0.0220)	June 1973
January 1934 – June 1973	10.024 (0.0563)	4.394 (0.0792)	3.054 (0.0544)	October 1969

The *Sup*-W_T, *Ave*-W_T, and *Exp*-W_T are the supremum, average, exponential average of the Wald statistics for testing the null hypothesis of no structural breaks at unknown change points. These tests of structural breaks and parameter instability are applied to the unrestricted model in equation (1). The numbers in parentheses are the *p*-values. The Max date in the last column is the point where the maximum value of the Wald statistic occurs during the time interval considered.

the maximum value of the statistic occurs as the single structural break. Since the *p*-value of the *Sup*-W_T statistic in the third period reaches its single trough on April 1981, we consider April 1981 as a structural break-date. The findings in Table 5 provide strong evidence in favor of a structural break at the beginning of 1981, when the interest rate begins to decline substantially. The results are robust across the *Sup*, *Ave*, and *Exp* tests since all *p*-values in the first row of Table 5 indicate a rejection of the null hypothesis of parameter stability at the 5% significance level.

To test if the model displays more structural breaks, we exclude the period after April 1981 from the effective sample and run the test again. The results are reported in the second row of Table 5. The test-statistics suggest that another structural break occurred in June 1973 when the Fed begins to implement the federal-fund operating procedure for the period spanning September 1972–October 1979. We also follow the same procedure to test for the existence of a third break. Fig. 2 seems to suggest that the third structural breakpoint is October 1969. However, based on the *p*-values in the third row of Table 5, we see that the *Sup*, *Ave*, and *Exp* tests statistics cannot reject the hypothesis of parameter stability during this sub-period.

5.3.2. The case of raw, aggregate interest-rate series

In Table 6, the goodness-of-fit tests indicate that most of the models are rejected by the raw interest-rate data for the sub-period from January 1934 to June 1973. We observe that the models with moderate

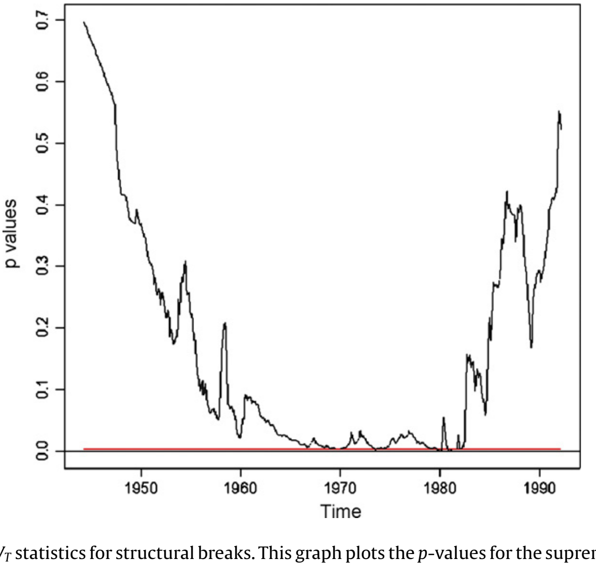

Fig. 2. The Andrew's *Sup*-W_T statistics for structural breaks. This graph plots the *p*-values for the supremum of the Wald statistics for the period from 1934 to 2002. The horizontal doubled-line represents the 5% *p*-values. At a conventional 5% level, one rejects the null hypothesis of no structural breaks for the values of the supremum that are below this horizontal line.

Table 6
GMM estimators and goodness-of-fit for the sub-period 1934–1973.

Model	α	β	σ^2	γ	χ^2
Unrestricted	0.0065 (0.545)	0.0042 (0.559)	0.0015 (0.0031)	0.579 (1.51e−06)	
Merton	0.0154 (0.0229)	0	0.0209 (2.01e−08)	0	23.657 (7.29e−06)
Vasicek	0.0104 (0.328)	0.0035 (0.626)	0.0209 (2.82e−08)	0	23.728 (1.11e−06)
CIR SR	0.0075 (0.474)	0.0037 (0.607)	0.0177 (7.10e−15)	0.5	0.485 (0.486)
Dothan	0	0	0.0039 (9.68e−14)	1.0	9.130 (0.0276)
GBM	0	0.0077 (0.0923)	0.0039 (1.41e−13)	1.0	6.482 (0.0391)
Brennan–Schwartz	0.0025 (0.810)	0.0067 (0.361)	0.0039 (1.76e−13)	1.0	6.472 (0.0109)
CIR VR	0	0	0.0007 (3.27e−11)	1.5	18.494 (0.0003)
CEV	0	0.0076 (0.0988)	0.0142 (0.0037)	0.590 (1.25e−06)	0.366 (0.545)

χ^2 is the test-statistic for the GMM overidentifying restrictions of each interest-rate model. The degree of freedom is the number of parameter restrictions that each nested model imposes on the unrestricted model. The numbers in parentheses are the p-values.

levels of volatility elasticity, the CEV ($\gamma = 0.590$) and CIR SR ($\gamma = 0.5$), outperform the other competing models. This result indicates that the volatility is less sensitive to the interest-rate level during this low interest-rate period. We also note that the estimates for volatility parameter σ are substantially lower than those for the full-sample estimates. In addition, the positive sign for the β estimates indicate lack of mean-reversion during this sub-period.

In Table 7, the p-values suggest that all the high-volatility elasticity models perform relatively well during the sub-period 1973–1981. In contrast, all the models with $\gamma < 1$ are rejected by the data. With a p-value of 0.269, we observe that the driftless CIR VR model provides the best fit for the interest-rate data, whereas the mean-reverting models, such as Vasicek and CIR SR, perform very poorly. The estimates for the γ parameter are highly significant and those for the drift coefficients are insignificant during this high volatility regime of 1973–1981. The negative, but statistically insignificant value for the estimate of the speed of adjustment parameter, β, in the unrestricted model provides very weak evidence of drift-induced mean-reversion.

In Table 8, we see that all the models, except Vasicek and CIR SR, provide a good fit for the sub-period 1981 to 2002. The estimate for γ is around 1.94 for both the unrestricted and CEV, which implies that the level effect is higher than in the other sub-periods. We see that most models with $\gamma \geq 1$ outperform those with $\gamma < 1$. In contrast to the high-volatility sub-period, we see however that not all the models with $\gamma < 1$ are rejected by the raw interest-rate data. Specifically, the Merton model with $\gamma = 0$ cannot be rejected at the conventional 5% level and even outperforms the mean-reverting Brennan–Schwartz model, which has relatively high volatility elasticity. This seemingly conflicting result can be explained by the fact that this sub-period is primarily characterized by (non-monotonically) declining interest rates rather then high volatilities. The sharp downward trend in the short-term interest rate can be accommodated by either high volatility elasticity or negative expected growth rate as, for example, in

Table 7
GMM estimators and goodness-of-fit for the sub-period 1973–1981.

Model	α	β	σ^2	γ	χ^2
Unrestricted	0.338 (0.206)	−0.0375 (0.363)	0.0012 (0.316)	1.451 (4.61e−10)	
Merton	0.0942 (0.0898)	0	0.236 (2.48e−05)	0	6.965 (0.0307)
Vasicek	0.218 (0.408)	−0.0182 (0.652)	0.243 (5.16e−05)	0	7.205 (0.0073)
CIR SR	0.345 (0.204)	−0.0336 (0.415)	0.0433 (1.42e−05)	0.5	6.416 (0.0113)
Dothan	0	0	0.0064 (8.78e−07)	1.0	6.555 (0.0875)
GBM	0	0.0133 (0.114)	0.0060 (7.27e−06)	1.0	4.171 (0.124)
Brennan–Schwartz	0.345 (0.199)	−0.0371 (0.368)	0.0072 (8.14e−06)	1.0	2.461 (0.0117)
CIR VR	0	0	0.0009 (6.318−07)	1.5	3.934 (0.269)
CEV	0	0.0134 (0.114)	0.0007 (0.298)	1.541 (2.94e−11)	1.487 (0.223)

χ^2 is the test-statistic for the GMM overidentifying restrictions of each interest-rate model. The degree of freedom is the number of parameter restrictions that each nested model imposes on the unrestricted model. The numbers in parentheses are the p-values.

Table 8
GMM estimators and goodness-of-fit for the sub-period 1981–2002.

Model	α	β	σ^2	γ	χ^2
Unrestricted	0.120 (0.307)	−0.0280 (0.213)	6.87e−05 (0.617)	1.945 (5.20e−06)	
Merton	−0.0270 (0.101)	0	0.0548 (7.85e−09)	0	4.553 (0.103)
Vasicek	−0.0268 (0.773)	−6.94e−05 (0.997)	0.0541 (1.21e−08)	0	5.061 (0.0245)
CIR SR	−0.0186 (0.841)	−0.0018 (0.920)	0.0107 (2.86−09)	0.5	4.692 (0.0303)
Dothan	0	0	0.0017 (4.75e−09)	1.0	5.332 (0.149)
GBM	0	−0.0054 (0.0911)	0.0018 (1.07e−09)	1.0	3.594 (0.165)
Brennan–Schwartz	0.0030 (0.974)	−0.0060 (0.738)	0.0018 (1.07e−09)	1.0	3.846 (0.0498)
CIR VR	0	0	0.0003 (2.94e−09)	1.5	4.240 (0.237)
CEV	0	−0.0052 (0.0983)	5.38e−05 (0.562)	1.942 (6.83e−06)	1.029 (0.310)

χ^2 is the test-statistic for the GMM overidentifying restrictions of each interest-rate model. The degree of freedom is the number of parameter restrictions that each nested model imposes on the unrestricted model. The numbers in parentheses are the p-values.

the Merton model ($\alpha = -0.0270$). In contrast, the drift-induced mean-reversion cannot capture these abrupt downward movements and are therefore not well suited for this third sub-period.

5.3.3. *The case of detrended interest-rate series*

In Table 9, the goodness-of-fit tests indicate that none of the mean-reverting models are rejected by the detrended interest-rate data for the sub-period from January 1934 to June 1973. The p-value ranks the Vasicek ($\gamma = 0$) first, CIR SR ($\gamma = 0.5$) second, and Brennan–Schwartz ($\gamma = 1$) third across the nine models. Within the non-reverting models, we also find that a higher value for γ harms the performance of the model. The p-value ranks the Merton ($\gamma = 0$) first, Dothan ($\gamma = 1$) second, and CIR VR ($\gamma = 1.5$) third within the non-reverting group. As for the diffusion function, we find that the estimates for the diffusion parameters, σ^2 and γ, are statistically insignificant for both the unrestricted and CEV models. These parameter estimates provides further evidence of the insignificant role of the drift specification in capturing the dynamics of the stationary component of the interest rates. In addition, the statistically negative sign for the β estimates provide strong evidence of mean-reversion during this sub-period.

In Table 10, the p-values suggest that all the mean-reverting models outperform the non-reverting models during the sub-period 1973–1981. The p-value of the goodness-of-fit statistic for the Brennan–Schwartz ($\gamma = 1$), CIR SR ($\gamma = 0.5$), and Vasicek ($\gamma = 0$) model increases sharply from 0.011, 0.011, and 0.007 to 0.550, 0.296, and 0.194, respectively, implying strong acceptance of those models. In contrast, to the first sub-period, we see that the level effect plays an important, albeit secondary, role in the performance of the mean-reverting group where mean-reverting models with high volatility elasticity outperform the rest of the mean-reverting models. However, there is considerable worsening in the performance of the non-reverting high volatility elasticity models. For example, the p-value of the goodness-of-fit statistic for the CIR VR ($\gamma = 1.5$) and CEV model decreases dramatically from 0.269

Table 9
GMM estimators and goodness-of-fit using detrended interest-rate series for the sub-period 1934–1973.

Model	α	β	σ^2	γ	χ^2
Unrestricted	−0.0085 (0.7999)	0.0017 (0.7972)	1.01e−8 (0.9948)	1.6206 (0.9732)	
Merton	7.96e−06 (0.9282)	0.0	3.02e−06 (1.01e−11)	0.0	4.3570 (0.1132)
Vasicek	0.0014 (0.0246)	−0.0673 (0.0241)	3.24e−06 (5.01e−13)	0.0	0.1506 (0.6979)
CIR SR	0.0014 (0.0271)	−0.0664 (0.0259)	0.00015 (0.0000)	0.5	0.5220 (0.4699)
Dothan	0.0	0.0	0.0057 (1.50e−11)	1.0	6.8500 (0.0768)
GBM	0.0	−0.0022 (0.5836)	0.0058 (1.12e−11)	1.0	6.4497 (0.0397)
Brennan–Schwartz	0.0013 (0.0330)	−0.0643 (0.0310)	0.0061 (1.07e−12)	1.0	2.9706 (0.0847)
CIR VR	0.0	0.0	0.2238 (2.17e−11)	1.5	9.599 (0.0222)
CEV	0.0	−0.0028 (0.4063)	0.0053 (0.8115)	0.8321 (0.7384)	7.0114 (0.0341)

χ^2 is the test-statistic for the GMM overidentifying restrictions of each interest-rate model. The degree of freedom is the number of parameter restrictions that each nested model imposes on the unrestricted model. The interest-rate data is detrended using 10 Chebishev polynomials filtering technique. The numbers in parentheses are the p-values.

Table 10
GMM estimators and goodness-of-fit using detrended interest-rate series for the sub-period 1973–1981.

Model	α	β	σ^2	γ	χ^2
Unrestricted	0.0143 (0.0020)	−0.3570 (0.0040)	0.3558 (0.7840)	1.3714 (0.0167)	
Merton	0.0009 (0.2086)	0.0	0.00004 (1.52e−06)	0.0	3.9562 (0.1383)
Vasicek	0.0117 (0.0077)	−0.2850 (0.0150)	0.00003 (0.0001)	0.0	1.6797 (0.1949)
CIR SR	0.0124 (0.0047)	−0.3071 (0.0093)	0.0010 (0.0004)	0.5	1.0882 (0.2968)
Dothan	0.0	0.0	0.0256 (0.00006)	1.0	5.6262 (0.1312)
GBM	0.0	0.0169 (0.3925)	0.0272 (0.00003)	1.0	4.8028 (0.0905)
Brennan–Schwartz	0.0136 (0.0026)	−0.3387 (0.0052)	0.0314 (0.0007)	1.0	0.3557 (0.5508)
CIR VR	0.0	0.0	0.6091 (0.0001)	1.5	5.7172 (0.1262)
CEV	0.0	0.0196 (0.3387)	0.0024 (0.9223)	0.6558 (0.6759)	4.9695 (0.0257)

χ^2 is the test-statistic for the GMM overidentifying restrictions of each interest-rate model. The degree of freedom is the number of parameter restrictions that each nested model imposes on the unrestricted model. The interest-rate data is detrended using 10 Chebishev polynomials filtering technique. The numbers in parentheses are the p-values.

and 0.223 to 0.126 and 0.0257, respectively. In addition, the significant negative sign for the parameter β estimates suggest that the short-term rate is mean-reverting during this sub-period.

In Table 11, we also see that the mean-reverting models provide the best fit for the sub-period 1981–2002. As in the second sub-period, the p-value ranks the Brennan–Schwartz ($\gamma = 1$) first, CIR SR ($\gamma = 0.5$) second, and Vasicek ($\gamma = 0$) third across the nine models. Within the non-reverting group, we find that a higher value for γ does not necessarily improve the performance of the model. For example, the CEV model with an estimate of 3.306 for γ has a lower p-value than the CIR VR ($\gamma = 1.5$). In addition, we see that for the unrestricted and CEV models the volatility elasticity decreases from 1.649 and 1.592 to 1.386 and 0.993 when we detrend the interest rate. In addition, the significant negative sign for the parameter β estimates suggest that the detrended short-term rate is mean-reverting during this sub-period. This result suggests that for the stationary series a mean-reverting drift with high elasticity is the most important feature of the interest-rate model after 1973.

Clearly, these results for stationary component contradict those for the aggregate interest-rate series. In particular, for the filtered interest-rate series, we find strong evidence of mean-reversion and moderate level effects, whereas for the aggregate series, we find that a driftless model with high volatility elasticity outperforms all the competing models. In reconciling these opposing results, we note that mean-reversion can be induced by both the drift and diffusion function. Conley et al. (1997) show that, in essence, mean-reversion is determined by the ratio between the drift and two times the diffusion function. The ability of the latter to generate mean-reversion depends primarily on the values of diffusion parameters and the current interest-rate level. The higher these values, the greater the pull towards the center of the distribution. For the aggregate interest-rate process, this implies that the diffusion function captures the mean-reverting behavior and high level effects simultaneously,

Table 11
GMM estimators and goodness-of-fit using detrended interest-rate series for the sub-period 1981–2002.

Model	α	β	σ^2	γ	χ^2
Unrestricted	0.0072 (0.0036)	−0.1181 (0.0045)	0.3243 (0.7991)	1.7572 (0.0138)	
Merton	0.0002 (0.4003)	0.0	0.00001 (0.0002)	0.0	5.8803 (0.0528)
Vasicek	0.0053 (0.0177)	−0.0851 (0.0223)	0.00001 (0.0007)	0.0	2.4215 (0.1196)
CIR SR	0.0057 (0.0098)	−0.0933 (0.0122)	0.0002 (0.0004)	0.5	1.5562 (0.2122)
Dothan	0.0	0.0	0.0047 (0.0001)	1.0	7.2818 (0.0634)
GBM	0.0	0.0027 (0.5530)	0.0045 (0.0004)	1.0	6.9826 (0.0304)
Brennan–Schwartz	0.0062 (0.0056)	−0.1013 (0.0073)	0.0046 (0.0003)	1.0	0.7463 (0.3876)
CIR VR	0.0	0.0	0.0726 (0.0003)	1.5	0.0003 (0.0222)
CEV	0.0	−0.0033 (0.4624)	0.4400 (0.9999)	3.3069 (0.9987)	7.9924 (0.0461)

χ^2 is the test-statistic for the GMM overidentifying restrictions of each interest-rate model. The degree of freedom is the number of parameter restrictions that each nested model imposes on the unrestricted model. The interest-rate data is detrended using 10 Chebishev polynomials filtering technique. The numbers in parentheses are the p-values.

whereas the drift parameterization is almost superfluous.[2] However, this critical role of the diffusion function disappears when we detrend the interest-rate series.

Our findings suggest that the evidence on high volatility elasticity and weak drift-induced mean reversion documented in the literature is primarily due to the nonlinear trend component of the short-term interest rate. The a priori assumption of stationarity when specifying and testing interest-rate models implicitly shifts all the nonstationarity in the innovations and makes the drift specification less relevant for the overall performance of the models.

6. Conclusion

In this paper, we have re-examined a wide variety of interest-rate models. We employ a wide range of unit-root tests to evaluate the stationary hypothesis of the short-term interest rate. Consistent with Bandi (2002), the unit-root hypothesis cannot be rejected when it is tested against the stationary and linear trend-stationary hypotheses. However, as in Bierens (1997), we find that the unit-root hypothesis is rejected when it is tested against the alternative of non-linear trend stationary.

We locate the stationary component of the interest-rate process using the Hamming (1973) and Bierens (1997) filtering approach and examine the ability of the interest-rate models to capture this component. The analysis shows that the performances of all mean-reverting models (even those models which assume homoscedastic interest-rate volatility) show dramatic improvement in the presence of stationarity. In contrast, we document a significant decline in the performance of the models with high volatility elasticity.

To verify these findings, we incorporate the trend component in the drift function of the CIR SR and the unrestricted CEV model. This modification substantially improves the performance of the CIR SR and reduces the level effect of the unrestricted model. Our results suggest that the evidence on high volatility elasticity documented in the literature is primarily due to the nonlinear trend component of the raw interest-rate series.

The sub-sample analysis shows that all the mean-reverting models outperform the non-reverting models. In addition, the sub-sample results show that both the magnitude and importance of volatility elasticity vary considerably across sub-periods. During the sub-period of low and slowly increasing interest rates, we see that volatility elasticity is relatively low and the homoscedastistic models outperform the hetroscedastic models. For the high-volatility regime and in the presence of sharply declining interest rates the volatility elasticity plays a secondary role in the models specification.

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We are very grateful to Herman Bierens and Gerald Whitney, for their comments and suggestions.

Appendix A.

As in Bierens (1997), we use the following auxiliary regression for the $t(m)$ and $A(m)$ unit-root tests,

$$\Delta r_t = \alpha r_{t-1} + \sum_{j=1}^{p} \phi_j \Delta r_{t-j} + \theta^T P_{t,n}^{(m)} + \varepsilon_t, \tag{12}$$

where $P_{t,n}^{(m)} = (P_{0,n}^*(t), P_{1,n}^*(t), \ldots . . P_{m,n}^*(t))^T$ are transformed Chebishev polynomials that are orthogonal to t. The $t(m)$ and $A(m)$ test statistics can be computed based on the following specifications:

$$t(m) = \frac{\hat{\alpha}}{s_{\hat{\alpha}}}, \tag{13}$$

[2] We find similar results when we examine the performance of the nonlinear drift specification proposed by Aït-Sahalia (1996).

$$A(m) = \frac{n\hat{\alpha}}{\left(1 - \sum_{j=1}^{p} \hat{\phi}_j\right)}. \tag{14}$$

The model-free $T(m)$ test is based on the following specification:

$$\Delta r_t = -\rho r_{t-1} + \lambda_0 + \rho\lambda_1 t + \theta^{(1,m)^T} P_{t,n}^{(1,m)} + u_t, \quad \rho \in \{0, 1\}, \tag{15}$$

where $\theta^{(1,m)} = (\theta_1, \theta_2 \ldots .\theta_m)^T$ and $P_{t,n}^{(1,m)} = (P_{1,n}^*(t), P_{2,n}^*(t), \ldots . P_{m,n}^*(t))^T$. Bierens suggests the following test statistic:

$$T(m) = n \frac{\left[\sum_{t=1}^{n} \Delta r_t P_{t,n}^{(1,m)} - \hat{\xi}_1 P_{n+1,n}^{(1,m)} - \hat{\xi}_2 P_{1,n}^{(1,m)}\right]^T \left[\sum_{t=1}^{n} \Delta r_t P_{t,n}^{(1,m)} - \hat{\xi}_1 P_{n+1,n}^{(1,m)} - \hat{\xi}_2 P_{1,n}^{(1,m)}\right]}{\sum_{t=1}^{n} (r_t - \hat{\theta}^T P_{t,n}^{(m)})^2} \tag{16}$$

where $\hat{\xi}_1$ and $\hat{\xi}_2$ are least squared coefficients of regressing $\sum_{t=1}^{n} \Delta r_t P_{k,n}^*(t)$ on $P_{k,n}^*(n+1)$ and $P_{k,n}^*(1)$ for $k = 1, \ldots, m$.

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