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A Posteriori Error Estimates for a Dual Mixed Finite Element Method of the Elasticity Problem

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Abstract
By the current paper, we introduce and analyze *a posteriori* error estimator for a new dual mixed finite element method of the elasticity problem. In this method, the tensor of the constraints is approximated by Brezzi-Douglas-Marini fields augmented by rotational of the conforming bubble. We will show that this error estimator is reliable and efficient. Proof of reliability is based on Helmholtz decompositions of generalized tensor fields. The efficiency is demonstrated by the use of classical inverse estimates. Moreover, this estimator is independent of the coefficient of compressibility and thus remains valid in the incompressible limit case.

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Dual Mixed FEM, A Posteriori Error Estimation, Elasticity Problem, Tensor of Constraints.

1. Introduction

Let us fix a bounded domain Ω with a polygonal boundary Γ. In this domain we consider isotropic elastic homogeneous material. Let (u_1, u_2) be the displacement field and $f = (f_1, f_2) \in \left[L^2(\Omega)\right]^2$ the body force by unit of mass. Thus, the displacement field $u = (u_1, u_2)$ satisfies the following equations and boundary conditions:

$$\begin{cases} -\mathrm{div}\,\sigma(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where the stress tensor $\sigma(u)$ is defined by:

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda tr(\varepsilon(u))\delta. \tag{1.2}$$

The positive constants μ and λ are called Lamé coefficients. We assume that : [4]

$$(\lambda,\mu) \in [\lambda_0, +\infty[\times [\mu_1,\ \mu_2]$$

where:

$$0 \langle \mu_1 \langle \mu_2 \qquad \text{and} \qquad \lambda_0 \rangle 0 .$$

As usual, ε (u) denotes the linearized strain tensor (i.e., $\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right)$) and δ the identity tensor. The classical variational formulation of the boundary value problem (1.1) is the following [4, 7]: Find $u \in \left[H_0^1(\Omega)\right]^2 = \left\{ v \in \left[H^1(\Omega)\right]^2, \quad v|_\Gamma = 0 \right\}$ such that:

$$\int_{\Omega} \left(2\mu\varepsilon(\mathrm{u}) \ : \varepsilon(\mathrm{v}) + \lambda \operatorname{tr} \varepsilon(\mathrm{u}) \operatorname{tr} \varepsilon(\mathrm{v})\right) \mathrm{dx} = \int_{\Omega} \mathrm{f.vdx}, \qquad \forall \mathrm{v} \in \left[\mathrm{H}_0^1(\Omega)\right]^2. \tag{1.3}$$

Problem (1.3) has a unique solution for every $\mathrm{f} \in \left[\mathrm{H}^{-1}(\Omega)\right]^2$ (cf. [4], corollary 11.2.22, p.285). Several works have already been made on some various mixed finite elements methods concerning *a priori* error estimates as well as *a posteriori* error estimators [1],[13]. In this article, we are concerned by the construction of an efficient and reliable *a posteriori* error estimator for the new dual mixed formulation introducing as new unknowns:

$$\sigma := 2\mu\varepsilon(u) + \lambda tr(\varepsilon(u))\delta, \qquad p := -\frac{1}{2} tr\sigma \qquad \text{and} \qquad \omega := \frac{1}{2} rot(u) := \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

and the spaces:

$$\Sigma := \left[\mathrm{H}(\mathrm{div},\Omega)\right]^2 = \left\{ \tau \in \left[\mathrm{L}^2(\Omega)\right]^{2\times 2}, \quad \operatorname{div}\tau \in \left[\mathrm{L}^2(\Omega)\right]^2 \right\},$$

$$M := \left[\mathrm{L}^2(\Omega)\right]^2 \times \mathrm{L}^2(\Omega) \times \mathrm{L}_0^2(\Omega),$$

where $\mathrm{L}_0^2(\Omega) = \left\{ q \in \mathrm{L}^2(\Omega), \quad \int_{\Omega} q dx = 0 \right\}.$

The spaces Σ and M are respectively endowed with the natural norms:

$$\|\tau\|_{\Sigma} := (\|\tau\|_{0,\Omega}^2 + \|\operatorname{div}\tau\|_{0,\Omega}^2)^{\frac{1}{2}}, \qquad \|(\mathrm{v},\theta,\mathrm{q})\|_{\mathrm{M}} := (\|\mathrm{v}\|_{0,\Omega}^2 + \|\theta\|_{0,\Omega}^2 + \|\mathrm{q}\|_{0,\Omega}^2)^{\frac{1}{2}},$$

where from now on the notation $\| \, . \, \|_{0,\Omega}$ (resp. (. , .)) means the $L^2(\Omega)$-norm ($L^2(\Omega)$-inner product) of matrix valued functions, vector valued functions or scalar functions according to the context. In the following, we suppose $\mathrm{f} \in \left[\mathrm{L}^2(\Omega)\right]^2$.

With these notations we recall that the dual mixed formulation of problem (1.1) consists in finding $(\sigma,(u,\omega,p))$ in $\Sigma \times M$ solution of:

$$\begin{cases} \frac{1}{2\mu}(\sigma,\tau) + (div\tau,u) + (as(\tau),\omega) + \alpha'(tr\tau,p) = 0 & \forall \tau \in \Sigma, \\ (div\sigma,\mathrm{v}) + (as(\sigma),\theta) + (tr\sigma,q) + 2(p,q) + (f,\mathrm{v}) = 0 & \forall (\mathrm{v},\theta,\mathrm{q}) \in M, \end{cases} \tag{1.4}$$

where $\alpha' = \dfrac{\lambda}{2\mu(\lambda+\mu)}$.

Here $(\sigma,\tau) = \int_{\Omega} \sigma : \tau \, \mathrm{dx}$, where $\sigma : \tau$ means the standard notation for the contraction of two tensors, i.e., $\sigma : \tau = \sum_{i,j=1}^{2} \sigma_{ij}\tau_{ij}$.

By the present approach, the symmetry of the strain tensor σ is relaxed by a Lagrange multiplier that is nothing else than the rotation ω. Problem (1.4) will be approximated by conforming finite element spaces $\Sigma_h \times M_h$ of $\Sigma \times M$ based on a triangulation T_h of the domain Ω from a regular family (regular in Ciarlet's sense [7]). The discrete problem has a unique solution $(\sigma_h,(u_h,\omega_h,p_h)) \in \Sigma_h \times M_h$. We then consider an efficient and reliable a posteriori error estimator of residual type for the errors $\varepsilon := \sigma - \sigma_h$, $\mathrm{r} := \mathrm{p} - \mathrm{p}_h$, $\mathrm{s} := \omega - \omega_h$ and $\mathrm{e} := \mathrm{u} - \mathrm{u}_h$.

Our analysis release on a residual error indicator η, which is based on residuals on each triangle $K \in T_h$ and jumps across the interelement boundaries $E \in \xi_h$. Our goal in this article is to prove reliability and efficiency of the indicator η uniformly in λ and h, in particular avoiding locking phenomena.

The proof of the reliability of our indicator is based on some generalised Helmholtz decomposition of tensor fields. Efficiency follows by using classical inverse estimates [15]; see section 5 for more details.

The schedule of this article is as following: section 2 recalls the discretization of our problem and we give some preliminaries and notations. In section 3, we establish some results on tensor fields like some particular Helmholtz decomposition. In section 4 we recall some standard tools, namely some inverse inequalities and interpolation error estimates for Clément's interpolant. We finish by establishing the efficiency and reliability of our error indicator η in section 5.

Finally, let us precise some notations that will be used subsequently. For any tensor fields $\tau = (\tau_{ij})_{1\le i,j\le 2} \in \left[H^1(\Omega)\right]^{2\times 2}$ and for any vector fields $v = (v_1, v_2) \in \left[H^1(\Omega)\right]^2$, we define:

$$tr\tau := \tau_{11} + \tau_{22},$$

$$\mathrm{div}(\tau) := \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2}, \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2}\right),$$

$$\mathrm{as}(\tau) := \tau_{21} - \tau_{12},$$

$$rot(\tau) := \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2}\right),$$

$$\mathrm{Curl}(v) := \begin{pmatrix} \frac{\partial v_1}{\partial x_2} & -\frac{\partial v_1}{\partial x_1} \\ \frac{\partial v_2}{\partial x_2} & -\frac{\partial v_2}{\partial x_1} \end{pmatrix},$$

$$\mathrm{rot}(v) := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2},$$

$$\mathrm{div}(v) := \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}.$$

The norm and semi-norm of the standard Sobolev space $H^1(\Omega)$ is denoted by $\|\,.\,\|_{1,\Omega}$ and $|\,.\,|_{1,\Omega}$, respectively. Finally, in order to avoid excessive use of constants, we will use the following notations: a ~ b and $a \prec b$ stand for $c_1 b \le a \le c_2 b$ and $a \le cb$, respectively, with positive constants c, c_1, c_2 independent of a, b, λ and h.

2. Discretization

2.1 Discretization of the domain Ω

The domain Ω is discredized by a family of triangulations $(T_h)_{h>0}$ made of triangles and regular in Ciarlet's sense (cf. [7]). Elements will be denoted by K and its edges are denoted by E. The set of edges of the triangulation will be denoted by ξ_h. Let x denote a nodal point, and let N be the set of all (internal and boundary) nodes of the mesh. The measure of an element or an edge is denoted by $|K|$ and $|E|$, respectively. For an edge E of an element K, introduce the outer normal vector by n. Furthermore, for each edge E we fix one of the two normal vectors and denote it by n_E. Introduce additionally the tangent vector $t = n^{\perp}$ such that it is oriented positively. Similarly set $t_E = n_E^{\perp}$. The jump of some (scalar or vector valued) function v across an edge E is then defined as:

$$[\![\mathrm{v}(y)]\!]_E := \lim_{\alpha \to 0^+} \mathrm{v}(y + \alpha n_E) - \mathrm{v}(y - \alpha n_E), \qquad y \in E.$$

As usual, let ω_K be the union of all elements having a common edge with K. Similarly let ω_E be the union of both elements having E as edge. By ω_{x} we denote the union of all elements having x as node.

2.2 Discrete Mixed Formulation

This section concerns the approximation of the dual mixed problem (1.4). For each fixed triangulation T_h, we introduce the finite dimensional spaces Σ_h and M_h defined by the following way:

$$\Sigma_h = \left\{ \tau_h \in \Sigma; \tau_h \,|_K \in [BDM_1(K)]^2 + \alpha \begin{pmatrix} rotb_K \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ rotb_K \end{pmatrix}, \alpha, \beta \in R, \quad \forall K \in T_h \right\}$$

and $M_h = \left\{ (v_h, \theta_h, q_h) \in M; v_h \,|_K \in [P_0(K)]^2,\ \theta_h \,|_K \in P_1(K) \ \ and \ \ q_h \,|_K \in P_1(K), \quad \forall K \in T_h \right\}$

where $BDM_1(K) = [\mathrm{P}_1(\mathrm{K})]^2$ is the Brezzi- Douglas-Marini element (cf. [5]), $\mathrm{b_K}$ denotes the bubble function for the actual element K and $\mathrm{rotb_K} = \left(\frac{\partial \mathrm{b_K}}{\partial x_2}, -\frac{\partial \mathrm{b_K}}{\partial x_1} \right)$. The discrete problem associated with (1.4) is to find $\sigma_h \in \Sigma_h$ and $(u_h, \omega_h, p_h) \in M_h$ such that:

$$\begin{cases} \frac{1}{2\mu}(\sigma_h, \tau_h) + (div\tau_h, u_h) + (as(\tau_h), \omega_h) + \alpha'(tr\tau_h, p_h) = 0 & \forall \tau_h \in \Sigma_h, \\ (div\sigma_h, v_h) + (as(\sigma_h), \theta_h) + (tr\sigma_h, q_h) + 2(p_h, q_h) + (f, v_h) = 0 & \forall (v_h, \theta_h, q_h) \in M_h. \end{cases} \tag{2.1}$$

We recall that this problem has a unique solution. The existence and the uniqueness of the solution of discrete problem (2.1) are consequences of the following results (see [3]): $\exists \beta^* \rangle 0$ independent of h, such that $\forall v_h \in \prod_{K \in T_h} [P_0(K)]^2$, $\forall \theta_h \in \prod_{K \in T_h} P_1(K)$,

$\displaystyle Sup_{\tau_h \in \Sigma_h} \frac{(div\tau_h, v_h) + (as(\tau_h), \theta_h)}{\|\tau_h\|_\Sigma} \geq \beta^* \|(v_h, \theta_h)\|_{0,\Omega}$ and the fact that the only solution to the homogeneous problem associated with (2.1) is zero.

The analysis of the problem (2.1) and *a priori* error estimates were best dealt with in Boualem [3]. Our goal in this article is to propose and analyze *a posteriori* error estimates for problem (2.1). Let us mention that an *a posteriori* error estimates for the problem (1.4) was developed and analysed by Cartensen, Causin and Sacco [6] but using a finite element similar to the finite element PEERS (see Arnold, Brezzi and Douglas [2]). One of the interests of the method proposed here is that the approximation of the rotational of the displacement field ω is discontinuous. This is important for the implementation of such a problem (2.1) through a hybrid form of the latter (for details, see Farhloul and Fortin [9] and Boualem [3]).

We close this section by introducing, for any bounded domain Ω in R^2 with Lipschitz boundary, the space:

$$\mathrm{H}(rot, \Omega) := \left\{ \tau \in \left[\mathrm{L}^2(\Omega)\right]^{2\times 2} \ ; \ rot(\tau) \in \left[\mathrm{L}^2(\Omega)\right]^2 \right\} .$$

We recall the following formula of integration by parts: for all $\rho \in H(\text{rot},\Omega)$ and for all $\varphi \in [H^1(\Omega)]^2$

$$\int_\Omega rot(\rho)\,.\,\varphi\,dx - \int_\Omega \rho : Curl\varphi\,dx = \langle \rho.t, \varphi\rangle_{H^{-\frac{1}{2}}(\Gamma),H^{\frac{1}{2}}(\Gamma)}\,. \tag{2.2}$$

3. Decomposition for Tensor Fields

For our further analysis, we require the following results on the decomposition of tensor fields which are essential for the subsequent proofs.

Proposition 3.1. Let $\tau \in \left[L^2(\Omega)\right]^{2\times 2}$. Then there exist $p \in \left[H_0^1(\Omega)\right]^2$ and $\varphi \in \left[H^1(\Omega)\right]^2$ such that:

$$\tau = \nabla p + Curl\varphi\,, \tag{3.1}$$

with the estimate:

$$\|\nabla p\|_{0,\Omega} + \|\nabla\varphi\|_{0,\Omega} \prec \|\tau\|_{0,\Omega}. \tag{3.2}$$

Proof. Let $p \in \left[H_0^1(\Omega)\right]^2$ be the unique solution of the following problem:

$$\int_\Omega (\tau - \nabla p):\nabla\psi\,dx = 0 \qquad \text{for all } \psi \in \left[H_0^1(\Omega)\right]^2. \tag{3.3}$$

Applying Green's formula to this problem, we obtain:

$$-\int_\Omega \text{div}(\tau - \nabla p)\,.\,\psi\,dx = 0 \qquad \text{for all } \psi \in \left[H_0^1(\Omega)\right]^2.$$

Then $\tau - \nabla p$ is divergence free in the sense of distributions. Applying Theorem 3.1, p 37 of [12] line by line to the tensor $\tau - \nabla p$, we conclude that there exists a function $\varphi \in \left[H^1(\Omega)\right]^2$ such that $\tau - \nabla p = \text{Curl}\varphi$. Now, let us prove the estimate (3.2). Taking as test function $\psi = p$ in (3.3), we obtain:

$$\|\nabla p\|_{0,\Omega}^2 = \int_\Omega \tau : \nabla p\,dx \le \|\tau\|_{0,\Omega}\|\nabla p\|_{0,\Omega}.$$

Thus, $\|\nabla p\|_{0,\Omega} \le \|\tau\|_{0,\Omega}$. On the other hand, we have:

$$\|\nabla\varphi\|_{0,\Omega} = \|\text{Curl}\varphi\|_{0,\Omega} = \|\tau - \nabla p\|_{0,\Omega} \le \|\tau\|_{0,\Omega} + \|\nabla p\|_{0,\Omega} \le 2\|\tau\|_{0,\Omega}.$$

Consequently we have proved (3.2).

Proposition 3.2. Let $\tau \in \left[L^2(\Omega)\right]^{2\times 2}$. Then there exist $Z \in \left[H_0^1(\Omega)\right]^2$ and $\psi \in \left[H^1(\Omega)\right]^2$ such that:

$$\tau = 2\mu\,\varepsilon(Z) + \lambda\,\text{tr}(\varepsilon(Z))\delta + \text{Curl}\psi\,, \tag{3.4}$$

with the estimate:

$$\|\varepsilon(Z)\|_{0,\Omega} + \|\nabla\psi\|_{0,\Omega} \prec \|\tau\|_{0,\Omega}. \tag{3.5}$$

Proof. Let $Z \in \left[H_0^1(\Omega)\right]^2$ be the unique solution of the following problem:

$$\int_\Omega \left(2\mu\varepsilon(Z) + \lambda\,\text{tr}(\varepsilon(Z))\,\delta\right):\nabla\varphi\,dx = \int_\Omega \tau : \nabla\varphi\,dx \;\; \text{for all } \varphi \in \left[H_0^1(\Omega)\right]^2. \tag{3.6}$$

This last equation implies that $\tau - 2\mu\varepsilon(Z) - \lambda\,\text{tr}(\varepsilon(Z))\delta$ is a divergence free tensor in the sense of distributions. Applying Theorem 3.1, p 37, of [12] line by line to the tensor $\tau - 2\mu\varepsilon(Z) - \lambda\,\text{tr}(\varepsilon(Z))\delta$, we conclude that there exists a function $\psi \in \left[H^1(\Omega)\right]^2$ such that:

$\tau - 2\mu\varepsilon(Z) - \lambda\, \mathrm{tr}(\varepsilon(Z))\delta = \mathrm{Curl}\psi$. Now, let us prove the estimate (3.5). Taking as test function φ = Z in (3.6), we obtain $2\mu\|\varepsilon(Z)\|_{0,\Omega}^2 + \lambda\|\mathrm{div}Z\|_{0,\Omega}^2 = (\tau, \nabla Z)$.

By Cauchy-Schwarz's and Korn's inequalities, we derive from the previous equation:

$$\|\varepsilon(Z)\|_{0,\Omega} \prec \|\tau\|_{0,\Omega}.$$

By Lemma 3.4 [10], there exists $v \in \left[H_0^1(\Omega)\right]^2$ such that:

$$\mathrm{div}(v) = \mathrm{div}Z$$

and

$$\|\nabla v\|_{0,\Omega} \prec \|\mathrm{div}Z\|_{0,\Omega}.$$

Equation (3.6) with φ = v yields:

$$\lambda\|\mathrm{div}Z\|_{0,\Omega}^2 = -2\mu\int_\Omega \varepsilon(Z) : \nabla v\, dx + \int_\Omega \tau : \nabla v\, dx$$

$\prec 2\mu\|\varepsilon(Z)\|_{0,\Omega}\|\nabla v\|_{0,\Omega} + \|\tau\|_{0,\Omega}\|\nabla v\|_{0,\Omega} \prec \|\tau\|_{0,\Omega}\|\mathrm{div}Z\|_{0,\Omega}$ by the above bound on $\|\varepsilon(Z)\|_{0,\Omega}$.

Thus:

$$\lambda\|\mathrm{div}Z\|_{0,\Omega} \prec \|\tau\|_{0,\Omega}. \tag{3.7}$$

By triangular inequality, we get

$$\|\mathrm{Curl}\psi\|_{0,\Omega} \prec \|\tau\|_{0,\Omega} + \|\varepsilon(Z)\|_{0,\Omega} + \lambda\|\mathrm{div}Z\|_{0,\Omega} \prec \|\tau\|_{0,\Omega}.$$

Consequently, we have proved (3.5).

4. Analytical Tools

4.1 Bubble Functions, Extension Operator & Inverse Inequalities

For our further analysis we require standard bubble functions and extension operators that satisfy certain properties recalled here for the sake of completeness.

We need two types of bubble functions, namely b_K and b_E associated with an element K and an edge E, respectively. Denoting by $\lambda_{x_i}^K, x_i \in N \cap \partial K, i = 1, 2, 3,$ the barycentric coordinates of K and by $x_i^E \in N \cap E$, i = 1, 2, the vertices of the edge E⊂∂K we define:

$$b_K = 27\lambda_{x_1}^K\lambda_{x_2}^K\lambda_{x_3}^K \quad \text{and} \quad b_E = 4\lambda_{x_1^E}^{K_i}\lambda_{x_2^E}^{K_i} \text{ if } x \in K_i \text{ (i=1, 2)},$$

where K_1 and K_2 are the adjacent triangles to the edge E. One recalls that:

$$b_K = 0 \text{ on } \partial K, \qquad b_E = 0 \text{ on } \partial\omega_E, \qquad \|b_K\|_{\infty,T} = \|b_E\|_{\infty,\omega_E} = 1.$$

For an edge E⊂∂K using temporarily the local coordinates system (x, y) such that E is included into the x-axis, then the extension $F_{ext}(v_E)$ of $v_E \in C(E)$ to $C(\omega_E)$ is defined by $F_{ext}(v_E)(x, y) = v_E(x)$. Now we recall the so-called inverse inequalities that are proved using classical scaling techniques [15].

Lemma 4.1. (Inverse inequalities) Let $v_K \in P_{k_0}(K)$ and $v_E \in P_{k_1}(E)$, for some nonnegative integers k_0 and k_1. Then the following inequalities hold, the inequality constants depending on the polynomial degree k_0 or k_1 but not on K, E or v_K, v_E:

$$\left\|v_K b_K^{\frac{1}{2}}\right\|_{0,K} \sim \|v_K\|_{0,K} \tag{4.1}$$

$$\|\nabla(v_K b_K)\|_{0,K} \prec h_K^{-1}\|v_K\|_{0,K} \tag{4.2}$$

$$\left\| v_E b_E^{\frac{1}{2}} \right\|_{0,K} \sim \| v_E \|_{0,E} \tag{4.3}$$

$$\| F_{ext}(v_E) b_E \|_{0,K} \prec h_E^{\frac{1}{2}} \| v_E \|_{0,E}, \quad \forall K \subset \omega_E \tag{4.4}$$

$$\| \nabla (F_{ext}(v_E) b_E) \|_{0,K} \prec h_E^{\frac{1}{2}} h_K^{-1} \| v_E \|_{0,E}, \quad \forall K \subset \omega_E. \tag{4.5}$$

4.2 Clément Interpolation

For the analysis we require some interpolation operator that maps a function from $H^1(\Omega)$ to the usual space $S(\Omega, T_h)$ made of continuous and piecewise linear functions on the triangulation. Hence, Lagrange interpolation is unsuitable, but Clément like interpolant is more appropriate. Recall that the nodal basis function $\phi_x \in S(\Omega, T_h)$ associated with a node x is uniquely determined by the condition:

$$\phi_x(y) = \delta_{x,y} \qquad \text{for all } y \in N.$$

Next, the Clément interpolation operator will be defined via the basis functions $\phi_x \in S(\Omega, T_h)$.

Definition. (Clément interpolation operator) We define the Clément interpolation operator $I_{Cl} : H^1(\Omega) \to S(\Omega, T_h)$ by:

$$I_{Cl} v := \sum_{x \in N} \frac{1}{|\omega_x|} \left(\int_{\omega_x} v \right) \phi_x .$$

Finally we may state the interpolation estimates (for the proof we refer to [8]).

Lemma 4.2. (Clément interpolation estimates) Let $v \in H^1(\Omega)$. If the triangulation T_h is regular, then for any $E \in \xi_h$ and for any $K \in T_h$ it holds:

$$h_E^{-\frac{1}{2}} \| v - I_{Cl} v \|_{0,E} \prec \| \nabla v \|_{\omega_E}, \tag{4.6}$$

$$h_K^{-1} \| v - I_{Cl} v \|_{0,K} \prec \| \nabla v \|_{\omega_K}. \tag{4.7}$$

5. A Posteriori Error Estimation for Mixed FEM

We propose an a posteriori error estimator for the errors $\varepsilon := \sigma - \sigma_h, r := p - p_h, s := \omega - \omega_h$ and $e := u - u_h$ for our method. The local estimator accounts for the residuals on the triangle K and the jumps across the edges $E \subset \partial K$. By the following, we denote the jump in the tangential direction of a discrete tensor ρ_h by $\left[\left| \rho_h . t_E \right| \right]_E$. For any $K \in T_h$, the local residual error estimator η_K is defined by:

$$\begin{aligned} \eta_K^2 = {} & \| f - P_h^0 f \|_{0,K}^2 + \| as(\sigma_h) \|_{0,K}^2 + \left\| p_h + \frac{1}{2} tr \sigma_h \right\|_{0,K}^2 + h_K^2 \| \sigma_h + 2\mu\alpha' p_h \delta + 2\mu\omega_h \chi \|_{0,K}^2 + \\ & h_K^2 \| rot(\sigma_h + 2\mu\alpha' p_h \delta + 2\mu\omega_h \chi) \|_{0,K}^2 + \sum_{E \subset \partial K} h_E \left\| \left[\left| (\sigma_h + 2\mu\alpha' p_h \delta + 2\mu\omega_h \chi) \ . \ t_E \right| \right]_E \right\|_{0,E}^2 . \end{aligned} \tag{5.1}$$

Here P_h^0 denotes the L^2-orthogonal projection onto the space of piecewise constant function on the triangulation T_h. As in [11], χ denotes the constant tensor field $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The global residual error estimator is simply defined by:

$$\eta^2 = \sum_{K \in T_h} \eta_K^2.$$

Let us observe in the right hand of (5.1) that $f - P_h^0 f$ is the residual of –div(σ)=f, and that the two following terms have zero as their analogues for the exact solution. $h_K^2 \|\sigma_h + 2\mu\alpha' p_h \delta + 2\mu\omega_h \chi\|_{0,K}^2$ is negligible compared to these two terms.
Also $\mathrm{rot}(\sigma_h + 2\mu\alpha' p_h \delta + 2\mu\omega_h \chi)$ and $\left[\!\left[(\sigma_h + 2\mu\alpha' p_h \delta + 2\mu\omega_h \chi) \cdot t_E\right]\!\right]_E$ have corresponding terms zero for the exact solution since $\sigma + 2\mu\alpha' p\delta + 2\mu\omega\chi = 2\mu\nabla u$.

5.1 Proof of the Reliability of Estimator

We begin with the estimate for the error $\varepsilon := \sigma - \sigma_h$. Let us point out that all the estimates that will be established are independent of the lame coefficient λ for $\lambda \rangle \lambda_0$.

Proposition 5.1. The following estimate holds:

$$\|\varepsilon\|_{0,\Omega} \prec \eta.$$

Proof. Proposition 3.2 implies the existence of $Z \in \left[H_0^1(\Omega)\right]^2$ and $\psi \in \left[H^1(\Omega)\right]^2$ such that:

$$\varepsilon + \frac{1}{2} as(\sigma_h)\chi = 2\mu\varepsilon(Z) + \lambda\, tr(\varepsilon(Z))\delta + \mathrm{Curl}\psi. \tag{5.2}$$

Moreover, the following estimate holds:

$$\|\varepsilon(Z)\|_{0,\Omega} + \|\nabla\psi\|_{0,\Omega} \prec \left\|\varepsilon + \frac{1}{2} as(\sigma_h)\chi\right\|_{0,\Omega}. \tag{5.3}$$

It follows from $\varepsilon := \sigma - \sigma_h$ and identity (5.2) that $\mathrm{Curl}\psi$ is a symmetric tensor fields that implies that divψ=0. By triangular inequality we have:

$$\|\varepsilon\|_{0,\Omega} \le \left\|\varepsilon + \frac{1}{2} as(\sigma_h)\chi\right\|_{0,\Omega} + \left\|\frac{1}{2} as(\sigma_h)\chi\right\|_{0,\Omega} = \left\|\varepsilon + \frac{1}{2} as(\sigma_h)\chi\right\|_{0,\Omega} + \frac{1}{\sqrt{2}}\|as(\sigma_h)\|_{0,\Omega}. \tag{5.4}$$

In view of the definition of the error estimator η, it suffices to bound $\left\|\varepsilon + \frac{1}{2} as(\sigma_h)\chi\right\|_{0,\Omega}$. The above decomposition allows to write:

$$\left\|\varepsilon + \frac{1}{2} as(\sigma_h)\chi\right\|_{0,\Omega}^2 = \left(\varepsilon + \frac{1}{2} as(\sigma_h)\chi, 2\mu\varepsilon(Z) + \lambda tr(\varepsilon(Z))\delta + Curl\,\psi\right)$$

$$= (\varepsilon, 2\mu\varepsilon(Z) + \lambda tr(\varepsilon(Z))\delta + Curl\,\psi) + \left(\frac{1}{2} as(\sigma_h)\chi, 2\mu\varepsilon(Z) + \lambda tr(\varepsilon(Z))\delta + Curl\,\psi\right)$$

$$= (\varepsilon, 2\mu\varepsilon(Z) + \lambda tr(\varepsilon(Z))\delta + Curl\,\psi)$$

since $2\mu\varepsilon(Z) + \lambda tr(\varepsilon(Z))\delta + Curl\,\psi$ is a symmetric tensor field. Then:

$$\left\|\varepsilon + \frac{1}{2} as(\sigma_h)\chi\right\|_{0,\Omega}^2 = 2\mu(\varepsilon, \varepsilon(Z)) + \lambda(tr\varepsilon, tr(\varepsilon(Z))) + (\varepsilon, Curl\,\psi) \tag{5.5}$$

To transform the last term of the right-hand side, let us consider $\psi_h := I_{Cl}\psi$. By the first equality of the continuous problem (1.4) with τ = Curl $\psi \in \Sigma$ and the fact that divψ=0, we get $(\sigma, \mathrm{Curl}\psi) = -2\mu\alpha'(\mathrm{tr}(\mathrm{Curl}\psi), \mathrm{p})$. Thus:

$$(\varepsilon, Curl\,\psi) = -2\mu\alpha'(tr(Curl\,\psi), p) + (\sigma_h, Curl(\psi_h - \psi)) - (\sigma_h, Curl\,\psi_h). \tag{5.6}$$

For the second term of the right-hand side of the last equality, taking as a test function $\tau_h = \mathrm{Curl}\psi_h \in \Sigma_h$ in the first equation of (2.1), we get:

$$(\sigma_h, Curl\,\psi_h) = -2\mu(div\,\psi_h, \omega_h) - 2\mu\alpha'(Curl\,\psi_h, p_h\delta)$$

Combined with the previous equality and (5.6), we obtain:

$$(\varepsilon, Curl\,\psi) = -2\mu\alpha'(tr(Curl\,\psi), p) + (\sigma_h, Curl(\psi_h - \psi)) + 2\mu(div\,\psi_h, \omega_h) + 2\mu\alpha'(Curl\,\psi_h, p_h\delta).$$

Remembering that divψ=0, it follows that:

$$\begin{aligned}(\varepsilon, Curl\,\psi) &= (\sigma_h + 2\mu\alpha' p_h\delta, Curl(\psi_h - \psi)) + (2\mu\alpha' p_h\delta, Curl\,\psi) - 2\mu\alpha'(tr(Curl\,\psi), p) + \\ &\quad 2\mu(div(\psi_h - \psi), \omega_h) \\ &= (\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi, Curl(\psi_h - \psi)) + 2\mu\alpha'(tr(Curl\,\psi), p_h - p). \end{aligned} \tag{5.7}$$

Now we will focus on the second term of the right-hand of (5.5). By using (5.2), we obtain

$$tr(\varepsilon(Z)) = \frac{1}{2(\mu+\lambda)}(tr(\varepsilon) - tr(Curl(\psi))).$$

Since $\frac{\lambda}{2(\mu+\lambda)} = \mu\alpha'$ and since $tr\varepsilon = tr\sigma - tr\sigma_h = -2p - tr\sigma_h$, we deduce that:

$$\lambda(tr\varepsilon, tr\varepsilon(Z)) = \mu\alpha'\|tr\varepsilon\|_{0,\Omega}^2 + 2\mu\alpha'\left(p + \frac{1}{2}tr\sigma_h, tr(Curl\,\psi)\right).$$

Using this last identity and (5.7) into (5.5) we finally obtain:

$$\begin{aligned}\left\|\varepsilon + \frac{1}{2}as(\sigma_h)\chi\right\|_{0,\Omega}^2 &= (\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi, Curl(\psi_h - \psi)) + \\ &\quad 2\mu\alpha'\left(p_h + \frac{1}{2}tr\sigma_h, tr(Curl\psi)\right) + \\ &\quad 2\mu(\varepsilon, \varepsilon(Z)) + \mu\alpha'\|tr(\varepsilon)\|_{0,\Omega}^2. \end{aligned} \tag{5.8}$$

We now estimate separately the three terms of the right-hand side of (5.8). For the first, by using Green's formula and the fact that ψ and ψ_h are continuous through the edges we get:

$$\begin{aligned}(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi, Curl(\psi_h - \psi)) &= \sum_{K\in T_h}\int_K rot(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi).(\psi_h - \psi)\,dx - \\ &\quad \sum_{E\in\xi_h}\int_E \left[\left|(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi).t_E\right|\right]_E.(\psi_h - \psi)\,ds. \end{aligned}$$

Besides, applying Cauchy-Schwarz's inequality and Lemma 4.2 we obtain:

$$\begin{aligned}(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi, Curl(\psi_h - \psi)) &\prec \left[\sum_{K\in T_h} h_K^2\|rot(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi)\|_{0,K}^2\right]^{\frac{1}{2}}\left[\sum_{K\in T_h}\|\nabla\psi\|_{\omega_K}^2\right]^{\frac{1}{2}} + \\ &\quad \left[\sum_{E\in\xi_h} h_E\left\|\left[\left|(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi).t_E\right|\right]_E\right\|_{0,E}^2\right]^{\frac{1}{2}}\left[\sum_{E\in\xi_h}\|\nabla\psi\|_{\omega_E}^2\right]^{\frac{1}{2}}. \end{aligned}$$

According to (5.3) we arrive at the estimate:

$$\left(\sigma_h + 2\mu\alpha' p_h \delta + 2\mu\omega_h \chi, Curl(\psi_h - \psi)\right) \prec \eta \left\| \varepsilon + \frac{1}{2} as(\sigma_h)\chi \right\|_{0,\Omega}. \qquad (5.9)$$

For the second term of the right-hand side of (5.8), using the discrete Cauchy-Schwarz's inequality and the estimate (5.3), we obtain:

$$2\mu\alpha'\left(p_h + \frac{1}{2} tr\sigma_h, tr(Curl\psi) \right) \prec \left[\sum_{K \in T_h} \int_K \left(tr(Curl(\psi))\right)^2 dx \right]^{\frac{1}{2}} \left[\sum_{K \in T_h} \int_K \left(p_h + \frac{1}{2} tr\sigma_h \right)^2 dx \right]^{\frac{1}{2}}$$

$$\prec \eta \left\| \varepsilon + \frac{1}{2} as(\sigma_h)\chi \right\|_{0,\Omega} \qquad (5.10)$$

For the third term of the right-hand side of (5.8) using the fact that:

$\varepsilon(Z) = \nabla Z - \frac{1}{2}(rotZ)\chi$ and with the help of the green's formula, we get:

$$\int_\Omega \varepsilon : \varepsilon(Z)\, dx = \int_\Omega \varepsilon : \nabla Z dx - \frac{1}{2}\int_\Omega \varepsilon : (rotZ)\, \chi dx = -\int_\Omega div\varepsilon.\ Z dx - \frac{1}{2}\int_\Omega as(\varepsilon)\ (rotZ)\, dx.$$

Since $div\varepsilon = div\sigma - div\sigma_h = -(f - P_h^0 f)$ and since $as(\varepsilon) = -as(\sigma_h)$, we deduce that:

$$\int_\Omega \varepsilon : \varepsilon(Z)\, dx = \int_\Omega \left(f - P_h^0 f\right) .\ Z dx + \frac{1}{2}\int_\Omega as(\sigma_h)\ (rotZ)\, dx$$

$$\prec \left(\sum_{K \in T_h} \left\| f - P_h^0 f \right\|_{0,K}^2 \right)^{\frac{1}{2}} \|Z\|_{0,\Omega} + \frac{1}{2} \left(\sum_{K \in T_h} \|as(\sigma_h)\|_{0,K}^2 \right)^{\frac{1}{2}} \|rot(Z)\|_{0,\Omega}^2.$$

By Korn's and Poincaré's inequalities and the estimate (5.3), we obtain:

$$\|rot(Z)\|_{0,\Omega} \prec \|\nabla Z\|_{0,\Omega} \prec \|\varepsilon(Z)\|_{0,\Omega} \prec \left\| \varepsilon + \frac{1}{2} as(\sigma_h) \right\|_{0,\Omega}, \qquad (5.11)$$

$$\|Z\|_{0,\Omega} \prec \|\nabla Z\|_{0,\Omega} \prec \|\varepsilon(Z)\|_{0,\Omega} \prec \left\| \varepsilon + \frac{1}{2} as(\sigma_h) \right\|_{0,\Omega}. \qquad (5.12)$$

Then:

$$\int_\Omega \varepsilon : \varepsilon(Z) dx \prec \eta \left\| \varepsilon + \frac{1}{2} as(\sigma_h)\chi \right\|_{0,\Omega}.$$

From (5.8), (5.9), (5.10) and this last bound, we obtain:

$$\left\| \varepsilon + \frac{1}{2} as(\sigma_h)\chi \right\|_{0,\Omega}^2 \prec \eta \left\| \varepsilon + \frac{1}{2} as(\sigma_h)\chi \right\|_{0,\Omega} + \mu\alpha' \|tr\varepsilon\|_{0,\Omega}^2.$$

By triangular inequality and the fact that $2\mu\alpha' \le 1$, we have:

$$\left\| \varepsilon + \frac{1}{2} as(\sigma_h)\chi \right\|_{0,\Omega}^2 \prec \eta \left(\|\varepsilon\|_{0,\Omega} + \|as(\sigma_h)\|_{0,\Omega} \right) + \frac{1}{2}\|tr\varepsilon\|_{0,\Omega}^2 \prec \eta \|\varepsilon\|_{0,\Omega} + \eta^2 + \frac{1}{2}\|tr\varepsilon\|_{0,\Omega}^2.$$

From (5.4), we get:

$\|\varepsilon\|_{0,\Omega} \prec \left\| \varepsilon + \frac{1}{2} as(\sigma_h)\chi \right\|_{0,\Omega} + \eta$. Thus $\|\varepsilon\|_{0,\Omega}^2 \prec \left\| \varepsilon + \frac{1}{2} as(\sigma_h)\chi \right\|_{0,\Omega}^2 + \eta^2.$

Then: $\|\varepsilon\|_{0,\Omega}^2 - \frac{1}{2}\|tr\varepsilon\|_{0,\Omega}^2 \prec \eta \|\varepsilon\|_{0,\Omega} + 2\eta^2.$

Since $\left\|\varepsilon^D\right\|_{0,\Omega}^2 = \left\|\varepsilon\right\|_{0,\Omega}^2 - \frac{1}{2}\left\|tr(\varepsilon)\right\|_{0,\Omega}^2$, where $\varepsilon^D := \varepsilon - \frac{1}{2}tr(\varepsilon)\delta$, we deduce that:
$\left\|\varepsilon^D\right\|_{0,\Omega}^2 \prec \eta\left\|\varepsilon\right\|_{0,\Omega} + 2\eta^2$.

Besides, by using the fact that $\int_\Omega tr\varepsilon \; dx = 0$ and according to [5], we obtain:
$\left\|\varepsilon\right\|_{0,\Omega} \prec \left\|\varepsilon^D\right\|_{0,\Omega} + \left\|div\varepsilon\right\|_{0,\Omega} \prec \left\|\varepsilon^D\right\|_{0,\Omega} + \eta$. Consequently, we obtain $\left\|\varepsilon\right\|_{0,\Omega}^2 \prec \eta\left\|\varepsilon\right\|_{0,\Omega} + 3\eta^2$.

Specifically, there exists $c \rangle 0$ such that:
$\left\|\varepsilon\right\|_{0,\Omega}^2 \le c\eta\left\|\varepsilon\right\|_{0,\Omega} + 3c\eta^2 \le \frac{1}{2}\left\|\varepsilon\right\|_{0,\Omega}^2 + \frac{1}{2}c^2\eta^2 + 3c\eta^2$.
Finally we get, $\left\|\varepsilon\right\|_{0,\Omega} \prec \eta$.

We turn now to bound the error $r := p - p_h$ by the error estimator η.

Proposition 5.2. The following estimate holds:

$$\left\|r\right\|_{0,\Omega} \prec \eta \tag{5.13}$$

Proof. From the second equations of the continuous problem (1.4), we get:
$\frac{1}{2}(tr\sigma, q) + (p, q) = 0 \;\; \forall q \in L_0^2(\Omega)$. Thus
$\frac{1}{2}(tr\sigma - tr\sigma_h, q) + (p - p_h, q) = -\left(p_h + \frac{1}{2}tr\sigma_h, q\right) \;\; \forall q \in L_0^2(\Omega)$.

Let $q = p - p_h \in L_0^2(\Omega)$ and by Cauchy-Schwarz's inequality we obtain:
$\left\|p - p_h\right\|_{0,\Omega}^2 \le \left(\frac{1}{2}\left\|tr\varepsilon\right\|_{0,\Omega} + \left\|p_h + \frac{1}{2}tr\sigma_h\right\|_{0,\Omega}\right)\left\|p - p_h\right\|_{0,\Omega}$.
Consequently we have proved (5.13).

Throughout the rest of this section, we use the notations $\beta_h := \sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi$ and $\beta := \sigma + 2\mu\alpha' p\delta + 2\mu\omega\chi$. Since Proposition 5.1 and Proposition 5.2 bounds respectively $\left\|\varepsilon\right\|_{0,\Omega} = \left\|\sigma - \sigma_h\right\|_{0,\Omega}$ and $\left\|r\right\|_{0,\Omega} = \left\|p - p_h\right\|_{0,\Omega}$ by a constant times η, it suffices to bound $\left\|\beta - \beta_h\right\|_{0,\Omega}$ in order to obtain an estimate for $\left\|\omega - \omega_h\right\|_{0,\Omega}$.

Lemma 5.3. The following estimate holds:

$$\left\|\beta - \beta_h\right\|_{0,\Omega} \prec \eta.$$

Proof. In view of Proposition 3.1, there exist $\mathrm{v} \in \left[H_0^1(\Omega)\right]^2$ and $\varphi \in \left[H^1(\Omega)\right]^2$ such that:

$$\frac{1}{2\mu}(\beta - \beta_h) = \nabla \mathrm{v} + Curl\varphi, \tag{5.14}$$

with the estimate:

$$\left\|\nabla \mathrm{v}\right\|_{0,\Omega} + \left\|\nabla\varphi\right\|_{0,\Omega} \prec \left\|\beta - \beta_h\right\|_{0,\Omega}. \tag{5.15}$$

By Green's formula, we have:

$$\|Curl\varphi\|_{0,\Omega}^2=\int_\Omega Curl\varphi:\left(\frac{1}{2\mu}(\beta-\beta_h)-\nabla \mathrm{v}\right)dx=\frac{1}{2\mu}\int_\Omega Curl\varphi:(\beta-\beta_h)dx$$
$$=-\frac{1}{2\mu}\int_\Omega Curl\varphi:\beta_h dx.$$

Let $\varphi_h=I_{Cl}\varphi$ be the Clément interpolation of φ. By the first equation of the discrete problem (2.1) with $\tau_h=Curl\varphi_h$, we have $(Curl\varphi_h,\beta_h)=0.$

Since φ_h and φ are continuous through the edges, we deduce that:

$$\|Curl\varphi\|_{0,\Omega}^2=\frac{1}{2\mu}\int_\Omega Curl(\varphi_h-\varphi):\beta_h dx$$
$$=\frac{1}{2\mu}\sum_{K\in T_h}\int_K (rot(\beta_h)).(\varphi_h-\varphi)dx-\frac{1}{2\mu}\sum_{E\in\xi_h}\int_E [\![\beta_h.t_E]\!]_E.(\varphi_h-\varphi)ds$$
$$\prec\sum_{K\in T_h}\|rot(\beta_h)\|_{0,K}h_K\|\nabla\varphi\|_{\omega_K}+\sum_{E\in\xi_h}\|[\![\beta_h.t_E]\!]_E\|_{0,E}h_E^{\frac{1}{2}}\|\nabla\varphi\|_{\omega_E}\quad\text{by (4.6) and (4.7).}$$
$$\prec\left\{\left(\sum_{K\in T_h}h_K^2\|rot\beta_h\|_{0,K}^2\right)^{\frac{1}{2}}+\left(\sum_{E\in\xi_h}h_E\|[\![\beta_h.t_E]\!]_E\|_{0,E}^2\right)^{\frac{1}{2}}\right\}\|Curl\varphi\|_{0,\Omega}.$$

Then:

$$\|Curl\varphi\|_{0,\Omega}\prec\eta. \tag{5.16}$$

Taking the symmetric parts in (5.2) and (5.14), we get:

$Sym(\varepsilon)=2\mu\ \varepsilon(Z)+\lambda\ tr(\varepsilon(Z))\ \delta+Curl\psi,$ as due to (5.2) $Curl(\psi)$ is a symmetric tensor field and:

$$Sym(\beta-\beta_h)=Sym(\varepsilon)+2\mu\alpha'(p-p_h)\delta=2\mu\ \varepsilon(v)+2\mu\ Sym(Curl(\varphi)).$$

Hence:

$$2\mu\ \varepsilon(Z-v)+\lambda\ tr(\varepsilon(Z))\delta+2\mu\alpha'(p-p_h)\delta=2\mu\ Sym(Curl\varphi)-Curl\psi.$$

Thus, we may estimate:

$$\begin{aligned}
&\|2\mu\ \varepsilon(Z-v)+\lambda\ tr(\varepsilon(Z))\delta+2\mu\alpha'(p-p_h)\delta\|_{0,\Omega}^2=\\
&\int_\Omega(2\mu\ Sym(Curl\varphi)-Curl\psi):(2\mu\ \varepsilon(Z-v)+\lambda\ tr(\varepsilon(Z))\delta+2\mu\alpha'(p-p_h)\delta)\,dx=\\
&\int_\Omega(2\mu\ Curl(\varphi)):(2\mu\varepsilon(Z-v)+\lambda tr(\varepsilon(Z))\delta+2\mu\alpha'(p-p_h)\delta)\,dx-\\
&\qquad\int_\Omega Curl\psi:(\lambda\ tr(\varepsilon(Z))\ \delta+2\mu\alpha'(p-p_h)\delta)\ dx
\end{aligned}\tag{5.17}$$

$$\le 2\mu\|Curl\varphi\|_{0,\Omega}\|2\mu\varepsilon(Z-\mathrm{v})+\lambda\ tr(\varepsilon(Z))\delta+2\mu\alpha'(p-p_h)\delta\|_{0,\Omega}+$$
$$\sqrt{2}\|Curl\psi\|_{0,\Omega}\|\lambda\ tr(\varepsilon(Z))+2\mu\alpha'(p-p_h)\|_{0,\Omega}$$
$$\le 2\mu^2\|Curl\varphi\|_{0,\Omega}^2+\frac{1}{2}\|2\mu\varepsilon(Z-\mathrm{v})+\lambda\ tr(\varepsilon(Z))\delta+2\mu\alpha'(p-p_h)\delta\|_{0,\Omega}^2+$$
$$\sqrt{2}\|Curl\psi\|_{0,\Omega}\left(\lambda\|divZ\|_{0,\Omega}+\|p-p_h\|_{0,\Omega}\right).$$

From (5.16), (5.3) and (3.7) (with $\tau=\varepsilon+\frac{1}{2}as(\sigma_h)\chi$) we get:

$\|2\mu\varepsilon(Z-\mathrm{v})+\lambda\, tr(\varepsilon(Z))\delta+2\mu\alpha'(p-p_h)\delta\|_{0,\Omega}\prec\eta$.

By Korn's and triangular inequalities we get:

$$\|\nabla \mathrm{v}\|_{0,\Omega}\prec\|\varepsilon(\mathrm{v})\|_{0,\Omega}\prec\|2\mu\;\varepsilon(Z-\mathrm{v})+\lambda\; tr(\varepsilon(Z))\delta+2\mu\alpha'(p-p_h)\delta\|_{0,\Omega}+$$
$$\|2\mu\;\varepsilon(Z)+\lambda\; tr(\varepsilon(Z))\delta+2\mu\alpha'(p-p_h)\delta\|_{0,\Omega}$$
$$\prec\eta+\|\varepsilon(Z)\|_{0,\Omega}+\|p-p_h\|_{0,\Omega}\prec\eta$$

by the estimate (3.5) with $\tau=\varepsilon+\frac{1}{2}as(\sigma_h)\chi$.

From (5.14), (5.16) and this last bound, we obtain by triangular inequality $\|\beta-\beta_h\|_{0,\Omega}\prec\eta$.

From the preceding Lemma, Proposition 5.1 and Proposition 5.2, we have immediately the following.

Proposition 5.4. The following bound holds:

$$\|\omega-\omega_h\|_{0,\Omega}\prec\eta.$$

It remains to prove that $\|u-u_h\|_{0,\Omega}\prec\eta$ in order to conclude that our estimator η is reliable.

Proposition 5.5. The following estimate holds:

$$\|u-u_h\|_{0,\Omega}\prec\eta\ .$$

Proof. Let $\tau\in\left[H^1(\Omega)\right]^{2\times 2}$ be such that (cf. [14]) (recall that $\mathrm{e}:=\mathrm{u}-\mathrm{u}_\mathrm{h}$)

$$\mathrm{div}\ \tau = \mathrm{e} \text{ in } \Omega,$$

with the property:

$$\|\tau\|_{1,\Omega}\prec\|e\|_{0,\Omega}\ . \tag{5.18}$$

Then, we may write:

$$\|e\|_{0,\Omega}^2=(u-u_h,div\,\tau)=(u,div\,\tau)-(u_h,div\,\tau).$$

In the first term of this right-hand side, we apply Green's formula and use the fact that u=0 on Γ, to obtain:

$$\|e\|_{0,\Omega}^2=-(\nabla u,\tau)-(u_h,div\tau)\ .$$

Now considering the global BDM interpolation operator $\prod_h$, setting $\tau_h=\prod_h\tau\in\Sigma_h$, and using the fact that u_h is constant on each triangle of the triangulation, we get:

$$\|e\|_{0,\Omega}^2=-\int_\Omega \tau:\nabla u dx-\int_\Omega u_h\ .\ div\,\tau_h dx$$
$$=-(\nabla u,\tau)+\frac{1}{2\mu}(\sigma_h,\tau_h)+(\tau_h,\omega_h\chi)+\alpha'(\tau_h,p_h\delta)$$ by the first equation of the discrete problem (2.1)
$$=-\frac{1}{2\mu}(\beta,\tau)+\frac{1}{2\mu}(\beta_h,\tau_h)$$
$$=-\frac{1}{2\mu}(\beta-\beta_h,\tau)+\frac{1}{2\mu}(\beta_h,\tau_h-\tau).$$

Furthermore, Cauchy-Schwarz inequality and the well-known error estimate:

$$\|\tau - \Pi_h \tau\|_{0,K} \prec h_K |\tau|_{1,K}$$

allow us to obtain:

$$\|e\|_{0,\Omega}^2 \prec \|\beta - \beta_h\|_{0,\Omega} \|\tau\|_{0,\Omega} + \sum_{K \in T_h} \|\beta_h\|_{0,K} h_K |\tau|_{1,K} \prec \left\{ \|\beta - \beta_h\|_{0,\Omega} + \left[\sum_{K \in T_h} h_K^2 \|\beta_h\|_{0,K}^2 \right]^{\frac{1}{2}} \right\} \|\tau\|_{1,\Omega}.$$

We conclude by the estimate (5.18).

5.2 Proof of the Efficiency of the Estimator

Recall further the notations $\beta = \sigma + 2\mu\alpha' p\delta + 2\mu\omega\chi$, $\beta_h = \sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi$, $\varepsilon := \sigma - \sigma_h$, $r := p - p_h$, $s := \omega - \omega_h$ and $e := u - u_h$. We treat separately the various contributions appearing in the estimator η.

Theorem 5.6. (Local lower error bound) For all $K \in T_h$ the following local lower error bound holds:

$$\eta_K \prec \left\| f - P_h^0 f \right\|_{0,K} + \|e\|_{0,K} + \|\varepsilon\|_{0,\omega_K} + \|s\|_{0,\omega_K} + \|r\|_{0,\omega_K}.$$

In order to prove this theorem we need the following Lemmas.

Lemma 5.7. The following estimate holds:

$$\left\| p_h + \frac{1}{2} tr\sigma_h \right\|_{0,K} \prec \|p - p_h\|_{0,K} + \|\sigma - \sigma_h\|_{0,K}.$$

Proof. Cauchy-Schwarz's inequality and the fact that $p + \frac{1}{2} tr\sigma = 0$ yield:

$$\left\| p_h + \frac{1}{2} tr\sigma_h \right\|_{0,K}^2 = -\int_K \left[(p - p_h) + \frac{1}{2} tr(\sigma - \sigma_h) \right] \left[p_h + \frac{1}{2} tr\sigma_h \right] dx$$

$$\prec \left(\|p - p_h\|_{0,K} + \|\sigma - \sigma_h\|_{0,K} \right) \left\| p_h + \frac{1}{2} tr\sigma_h \right\|_{0,K}.$$

This proves the Lemma.

Lemma 5.8. The following estimate holds:

$$h_K \|rot(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi)\|_{0,K} \prec \|\sigma - \sigma_h\|_{0,K} + \|p - p_h\|_{0,K} + \|\omega - \omega_h\|_{0,K}.$$

Proof. Inverse inequalities and Green's formula yield:

$$\|rot(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi)\|_{0,K}^2 \prec \int_K b_K \|rot(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi)\|^2 dx \qquad \text{by (4.1)}$$

$$= -\int_K rot(\beta - \beta_h) . b_K rot(\beta_h) dx = -\int_K (\beta - \beta_h) : Curl(b_K . rot(\beta_h)) dx$$

$$\le \|\beta - \beta_h\|_{0,K} \|Curl(b_K . rot(\beta_h))\|_{0,K} \prec \|\beta - \beta_h\|_{0,K} h_K^{-1} \|rot(\beta_h)\|_{0,K} \quad \text{by (4.2)}$$

$$\prec \left(\|\sigma - \sigma_h\|_{0,K} + \|p - p_h\|_{0,K} + \|\omega - \omega_h\|_{0,K} \right) h_K^{-1} \|rot(\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi)\|_{0,K}.$$

This proves the Lemma.

Lemma 5.9. For all $E \in \xi_h$ the following bound of the tangential jump error holds:

$$h_E^{\frac{1}{2}}\left\|\left[\left(\sigma_h+2\mu\alpha' p_h\delta+2\mu\omega_h\chi\right).t_E\right]_E\right\|_{0,E} \prec \|\sigma-\sigma_h\|_{0,\omega_E}+\|p-p_h\|_{0,\omega_E}+\|\omega-\omega_h\|_{0,\omega_E}.$$

Proof. Let us set $\psi_E := F_{ext}\left(\left[\beta_h.t_E\right]\right)b_E$, which belongs to $\left[H_0^1(\omega_E)\right]^2$.

As $\beta|_{\omega_E} \in \left[H(rot,\omega_E)\right]^2$, by integration by parts with ψ_E, we obtain:

$$\int_{\omega_E}\left(rot(\beta)\right).\psi_E dx-\int_{\omega_E}\beta : Curl\psi_E dx=\int_{\partial\omega_E}(\beta.t_E).\psi_E ds.$$

As rot(β) = 0 and $\psi_E\big|_{\partial\omega_E}=0$, this gives us:

$$\int_{\omega_E}\beta : Curl\psi_E dx=0.$$

For β_h we integrate elementarily and obtain:

$$\begin{aligned}\left\|\left[\beta_h.t_E\right]_E b_E^{\frac{1}{2}}\right\|_{0,E}^2 &=\int_E\left[\beta_h.t_E\right]_E.\psi_E ds=\sum_{K\subset\omega_E}\int_{\partial K}\beta_h.t.\psi_E ds\\ &=\sum_{K\subset\omega_E}\left[\int_K(rot(\beta_h)).\psi_E dx-\int_K\beta_h : Curl\,\psi_E dx\right]\\ &=\sum_{K\subset\omega_E}\left[\int_K(rot(\beta_h)).\psi_E dx+\int_K(\beta-\beta_h) : Curl\,\psi_E dx\right]\\ &\le\sum_{K\subset\omega_E}\|rot(\beta_h)\|_{0,K}\|\psi_E\|_{0,K}+\sum_{K\subset\omega_E}\|\beta-\beta_h\|_{0,K}\|Curl\,\psi_E\|_{0,K}.\end{aligned}$$

Lemma 5.8 and inverse inequalities (4.4), (4.5) lead to:

$$\begin{aligned}\left\|\left[\beta_h.t_E\right]_E b_E^{\frac{1}{2}}\right\|_{0,E}^2 \prec \sum_{K\subset\omega_E}&[h_K^{-1}\left(\|\sigma-\sigma_h\|_{0,K}+\|\omega-\omega_h\|_{0,K}+\|p-p_h\|_{0,K}\right)h_E^{\frac{1}{2}}\left\|\left[\beta_h.t_E\right]_E\right\|_{0,E}\\ &+\left(\|\sigma-\sigma_h\|_{0,K}+\|p-p_h\|_{0,K}+\|\omega-\omega_h\|_{0,K}\right)h_E^{\frac{1}{2}}h_K^{-1}\left\|\left[\beta_h.t_E\right]_E\right\|_{0,E}].\end{aligned}$$

The regularity of the triangulation enables us to bound $h_E^{\frac{1}{2}}h_K^{-1}\prec h_E^{-\frac{1}{2}}$ for all E⊂∂K with $K\in T_h$. Thus:

$$\left\|\left[\beta_h.t_E\right]_E b_E^{\frac{1}{2}}\right\|_{0,E}^2 \prec h_E^{-\frac{1}{2}}\left(\|\sigma-\sigma_h\|_{0,\omega_E}+\|\omega-\omega_h\|_{0,\omega_E}+\|p-p_h\|_{0,\omega_E}\right)\left\|\left[\beta_h.t_E\right]_E\right\|_{0,E}.$$

We conclude by using the equivalence (4.3).

Lemma 5.10. The following estimate holds:

$$h_K\|\sigma_h+2\mu\alpha' p_h\delta+2\mu\omega_h\chi\|_{0,K}\prec h_K\left(\|\sigma-\sigma_h\|_{0,K}+\|p-p_h\|_{0,K}+\|\omega-\omega_h\|_{0,K}\right)+\|u-u_h\|_{0,K}.$$

Proof. Recall that $\beta=\sigma+2\mu\alpha' p\delta+2\mu\omega\chi=2\mu\nabla u$. Now we have:

$$\|\sigma_h+2\mu\alpha' p_h\delta+2\mu\omega_h\chi\|_{0,K}^2\prec\left\|\beta_h b_K^{\frac{1}{2}}\right\|_{0,K}^2 \quad \text{by (4.1)}$$

$$
\begin{aligned}
&= -\int_K b_K(\beta-\beta_h):\beta_h dx + \int_K b_K\beta:\beta_h dx \\
&= -\int_K b_K(\beta-\beta_h):\beta_h dx + 2\mu\int_K b_K(\nabla(u-u_h)):\beta_h dx \\
&= -\int_K b_K(\beta-\beta_h):\beta_h dx - 2\mu\int_K (u-u_h).div(b_K\beta_h) dx \\
&\prec \|\beta-\beta_h\|_{0,K}\|\beta_h\|_{0,K} + 2\mu\|u-u_h\|_{0,K}\|div(b_K\beta_h)\|_{0,K} \\
&\prec \|\beta-\beta_h\|_{0,K}\|\beta_h\|_{0,K} + \|u-u_h\|_{0,K} h_K^{-1}\|\beta_h\|_{0,K} \quad \text{by (4.2)} \\
&\prec \left(\|\sigma-\sigma_h\|_{0,K} + \|p-p_h\|_{0,K} + \|\omega-\omega_h\|_{0,K} + h_K^{-1}\|u-u_h\|_{0,K}\right)\|\beta_h\|_{0,K}.
\end{aligned}
$$

In conclusion we have proved that:

$$
h_K\|\sigma_h + 2\mu\alpha' p_h\delta + 2\mu\omega_h\chi\|_{0,K} \prec h_K\left(\|\sigma-\sigma_h\|_{0,K} + \|p-p_h\|_{0,K} + \|\omega-\omega_h\|_{0,K}\right) + \|u-u_h\|_{0,K}.
$$

$\|as(\sigma_h)\|_{0,K}$ appearing in the estimator η_K is easily estimated as $as(\sigma_h) = -as(\sigma-\sigma_h)$, and the proof of Theorem 5.6 results from this and the sequence of Lemmas 5.7-5.10.

6. Conclusions

A new a posteriori error estimator for a dual mixed finite element method of the elasticity problem was introduced and analyzed. It was shown that this error estimator is reliable and efficient. Moreover, the estimator justifies an adaptive finite element scheme which refines a given grid only in regions where the error is relatively large. This method can treat natural boundary conditions, i.e., conditions of traction on a portion of the boundary of the domain and therefore it can be used for free boundary problems. On the other hand, the tensor of the constraints is naturally used in the coupling equations of linear elasticity with other equations such as the technological process of semiconductors. Therefore, this method would be best suited for this kind of problem.

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